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ON CORES AND PRIME IMPLICANTS OF TRUTH FUNCTIONS

W. V. QUINE, Harvard University

What is called a truth-functional formula in (alternational) *normal form* is built up of sentence letters 'p', 'q', etc., or their negations '$\bar{p}$', '$\bar{q}$', etc., or both, by using only the notations of conjunction and alternation, and in such a way as to subject alternations never to conjunction but only vice versa: thus '$pq \vee \bar{p}r\bar{s} \vee \bar{r}$'. This paper is concerned with the problem of reducing an arbitrary truth-functional formula to a shortest equivalent in normal form. Part of the content of my last previous paper on the subject* will be presented anew in an improved way, and a further theorem will be established. I shall not assume familiarity with my previous papers.

Let us sharpen our terminology. Sentence letters and their negations are called *literals*. Literals and conjunctions of them are called *fundamental formulas*, provided that none contains the same letter twice. Fundamental formulas and alternations of them are called *normal*, and are said to have those fundamental formulas as their *clauses*. On these definitions, a formula is convertible to normal form only if it is not self-contradictory; but there is no serious loss in setting aside the self-contradictory cases.

A *prime implicant* of a formula Φ is a fundamental formula that logically implies Φ but ceases to when deprived of any one literal. A normal formula will be called *uniliterally redundant* if it is equivalent to what remains of itself on dropping some one occurrence of a literal. Obviously then a normal formula is uniliterally redundant if and only if not all its clauses are prime implicants of it. Consequently, in particular,

(I) *Any shortest normal equivalent of a formula Φ will be an alternation of prime implicants of Φ.*

Ignoring tautologies along with self-contradictions, as will be convenient hereafter, one finds that

(II) *The prime implicants of a formula exist and are finite in number.*

To see that they exist, consider a formula Φ and any assignment of truth values to the letters of Φ that verifies Φ—say truth to 'p', 'q', and 's' and falsity to 'r'. The assignment determines a fundamental formula—'$pq\bar{r}s$', in this example—that implies Φ; and it, or part of it, is a prime implicant of Φ. To see further that the prime implicants of Φ are finite in number, we just reflect that they are bound to contain no letters foreign to Φ, since any such letter could be dropped without affecting the implication.

(III) *A formula is equivalent to the alternation of all its prime implicants.*

* A way to simplify truth functions, this MONTHLY, vol. 62, 1955, pp. 627–631. (Cited hereafter as WSTF.)

For, a formula Φ is implied by each of its prime implicants and hence by their alternation; and conversely every truth assignment that verifies Φ verifies some Φ-implying fundamental formula (*cf.* '$pq\bar{r}s$' above) and therewith some prime implicant (viz., that fundamental formula or part of it) and therewith the alternation of the prime implicants.

In view of (I) and (III), a formula can be transformed into a shortest normal equivalent in two stages: (A) transform it into the alternation of all its prime implicants and then (B) delete from that alternation the largest possible combination of jointly superfluous clauses. (A) can be accomplished by a technique due to Samson and Mills,* the explanation of which calls for two more definitions. If two fundamental formulas ϕ and ψ are opposed in exactly one letter (so that ϕ contains 'p' affirmatively, say, and ψ contains '$\bar{p}$'), then ϕ and ψ will be said to have as their *consensus* the formula which we get from the conjunction $\phi\psi$ by deleting the two opposed literals and any repetitions.§ If ϕ and ψ are fundamental formulas opposed in no letter, and all letters (hence all literals) of ψ are in ϕ, then ϕ will be said to (notationally) *subsume* ψ.†

Preparatory to the business of (A), we may suppose Φ put into normal form by familiar logical procedures. Now the discovery of Samson and Mills is that Φ, thus prepared, goes over into the alternation of its prime implicants if we persevere in these two equivalence transformations:

(i) If a clause subsumes another, drop the former.

(ii) Adjoin, as an additional clause, the consensus of two clauses (unless it subsumes a clause already present).

That (i)–(ii) are bound eventually to convert Φ into the alternation of just its prime implicants, and all of them (ignoring differences of order in a conjunction), is proved by proving the following four theorems.

(IV) *A normal formula remains susceptible to* (ii) *as long as some prime implicant of it is not a clause of it* (to within a permutation of a conjunction).

(V) *A normal formula remains susceptible to* (i) *or* (ii) *as long as some clause of it is not a prime implicant of it.*

(VI) *Normal formulas go into normal formulas under* (i) *and* (ii).

---

* Edward W. Samson and Burton E. Mills, Circuit minimization: algebra and algorithms for new Boolean canonical expressions, AFCRC Technical Report 54-21, April, 1954.

§ This definition of consensus is more liberal than that in WSTF, in that it counts ϕ as consensus of $\alpha\phi$ and $\bar{\alpha}$ (and of $\bar{\alpha}\phi$ and α). (i) below is consequently simpler than its counterpart in WSTF, and correspondingly for the proof of (IV) below. This liberalization of the definition is due essentially to Kurt Bing, On simplifying truth-functional formulas, J. Symbolic Logic, vol. 21, 1956, pp. 253 f; but note that I avoid his "void formula."

† My terminology has proved confusing. When ϕ subsumes ψ in my intended notational sense of having among its literals all those of ψ, then, precisely, ψ subsumes ϕ in a certain logical sense: ψ has among its verifying truth-value assignments all those of ϕ.

(VII)* *No normal formula is susceptible to* (i)–(ii) *without end.*

Proof of (IV). Let Φ be a normal formula and χ a prime implicant of it that differs (in more than order) from all clauses. Then also, being prime, χ subsumes no clause. So there is at least one fundamental formula, χ anyway, that has the three properties of (a) subsuming χ, (b) subsuming no clause of Φ, and (c) exhibiting no letters foreign to Φ (*cf.* end of proof of (II)). Moreover, there is an obvious limit to how long a fundamental formula having property (c) can be. So there is at least one fundamental formula, call it ψ, that has the three properties (a)–(c) and is exceeded in length by no fundamental formula having those properties. Yet ψ does not exhibit all letters of Φ; for, if it did, then, having also property (b) as it does, ψ would oppose every clause of Φ in one or another letter, and so not imply Φ; whereas actually ψ must imply Φ, having property (a). So ψ lacks some letter of Φ, say 'p'. Since ψ is a longest fundamental formula with the properties (a)–(c), $p\psi$ must lack one of those properties and so must $\bar{p}\psi$; yet obviously not (a) or (c); so (b). So there are clauses ϕ_1 and ϕ_2 of Φ such that $p\psi$ subsumes ϕ_1 and $\bar{p}\psi$ subsumes ϕ_2. But ψ, having the property (b), subsumes neither; so 'p' and '$\bar{p}$' must occur respectively in ϕ_1 and ϕ_2. Still ϕ_1 and ϕ_2 are not just 'p' and '$\bar{p}$', or Φ would be tautologous; nor are ϕ_1 and ϕ_2 opposed in letters other than 'p', since their further literals are common to ψ. So ϕ_1 and ϕ_2 have a consensus, say ϕ. Moreover ϕ subsumes no clause of Φ; for ψ subsumes ϕ, and ψ has the property (b). So ϕ can be added by (ii).

Proof of (V). Let Φ be a normal formula and ϕ any clause of it that is not a prime implicant. Then ϕ, implying Φ, subsumes some prime implicant ψ. If ψ is a clause of Φ (to within a permutation of a conjunction), we can apply (i) to drop ϕ; and otherwise we can apply (ii) in view of (IV).

Proof of (VI). Obvious.

Proof of (VII). Elimination of a clause ϕ by (i) depends on there surviving some clause ϕ' that ϕ subsumes; elimination of ϕ' in turn depends on there still surviving some clause ϕ'' that ϕ' (and hence ϕ) subsumes; and so on. Therefore, in view of the parenthetical part of (ii), no clause ϕ once dropped by (i) can ever be restored by (ii). But neither, in view of the parenthetical part of (ii), can a clause already present be reintroduced in duplicate by (ii). To sum up, (ii) can never introduce the same clause twice. But there are only finitely many different fundamental formulas for (ii) to introduce, since no letters foreign to Φ are drawn on. So the use of (ii) must terminate. Also the use of (i), being subtractive, obviously must terminate.

---

* This theorem and its proof, which I neglected to include in WSTF, are given here on the appreciated advice of the referee. Note that the parenthetical proviso in (ii), which may look dispensable in view of (i), is needed to assure (VII)—as is indeed remarked in effect in WSTF at the top of page 628.

So much for task (A), the conversion of a formula into the alternation of all its prime implicants. Toward the continuation task (B), that of determining what largest combination of the resulting clauses can be dropped as jointly superfluous, systematic methods have been developed by Ghazala.*

But a shortcoming of the whole (A)–(B) approach is that it depends on exhausting the prime implicants; for, as Rolf K. Müller remarked to me in 1955, their number can even exceed the total number of possible truth-value assignments when there are many variables. According to Fridshal,† this can happen after five variables. He cites a formula in nine variables which, by his computations, has 1698 prime implicants—whereas the total number of possible truth-value assignments is only 512. So a general technique is needed for finding an adequate minimum alternation of prime implicants without handling all prime implicants on the way.

A normal formula may be called *clausally redundant* if it is equivalent to what remains of itself on dropping one of its clauses of alternation; and *irredundant*, simply, if it is neither clausally nor uniliterally redundant. Obviously any shortest normal equivalent of a formula will be irredundant. Happily we can render a normal formula irredundant without regard to the totality of its prime implicants, simply ridding the formula of superfluous clauses and superfluous literals one by one. To see whether a clause ψ is superfluous in $\Phi \vee \psi$, we just check whether ψ implies Φ. Such implication cannot in general be recognized by mere notational subsumption, as can implication between fundamental formulas, but it can be checked very simply: we have merely to mark the literals of ψ as true and see whether Φ thereupon reduces to a tautology. Again, to see whether a literal ζ is superfluous in $\Phi \vee \zeta\psi$, we just check, in the same swift way, whether ψ implies $\Phi \vee \zeta$. For

(VIII) *ψ implies $\Phi \vee \zeta$ if and only if $\Phi \vee \zeta\psi$ is equivalent to $\Phi \vee \psi$.*

For, the conditional $\psi \supset . \Phi \vee \zeta$ is verifiably equivalent to the biconditional $\Phi \vee \zeta\psi \,.\equiv.\, \Phi \vee \psi$, and hence tautologous if and only if the biconditional is tautologous.

Unhappily, however, such reduction of a formula to *an* irredundant equivalent need not deliver a shortest. What it delivers may well lack some of the clauses of every shortest normal equivalent, and contain clauses shared by no shortest normal equivalent.

At the same time there commonly will be, given a formula Φ, certain clauses that are bound to appear in every irredundant equivalent of Φ, and hence in any shortest. The alternation of such clauses I call the *core* of Φ. Commonly also there will be, at the opposite extreme, prime implicants of Φ that are bound never to occur as clauses of irredundant equivalents of Φ. For a prime implicant

* M. J. Ghazala, Irredundant disjunctive and conjunctive forms of a Boolean function, I. B. M. Journal of Research and Development, vol. 1, 1957, pp. 171–176.

† R. Fridshal, The Quine algorithm, Summaries of Talks at the Summer Institute of Symbolic Logic (mimeographed), Cornell University, 1957, pp. 211 f.

of Φ to be thus absolutely superfluous, it is obviously sufficient that it imply the core of Φ; but I do not know whether this is necessary.

I shall explain a test whereby, without exhausting the prime implicants of a formula, we can identify its core. This done, we obviously can then quickly decide also, of a prime implicant, whether it implies the core (and is thus absolutely superfluous). Query: Given an irredundant formula, can all its absent prime implicants that are *not* absolutely superfluous be reached by the operation (ii) of iterated consensus-taking without interim retention of any absolutely superfluous ones? This, if true, would allow us to shortcut the (A)–(B) approach to the extent of leaving some of the prime implicants ungenerated in some cases; but I do not know it to be true, and anyway the saving would be limited to formulas with cores.

Guesses aside, there are cases where identifying the core of an irredundant formula settles the question of shortest normal equivalent without further ado. Thus if every clause belongs to the core, the formula is already as short as possible. If every clause but one belongs to the core, then again our simplification is good enough; the only possible improvement would be the negligible one of finding some shorter clause to take the place of the odd one.

But in the general case a technique of core identification has, alas, no evident bearing on our central problem: that of finding shortest normal equivalents without exhausting prime implicants. It seems worth communicating mainly for what it may contribute to one's understanding of the general workings of irredundant formulas.

Criterion, given an irredundant formula Φ, of whether a clause ϕ thereof belongs to the core: From Φ delete ϕ, also each clause that opposes ϕ in more than one letter, and finally all literals whose letters are in ϕ; what remains will be a tautology if and only if ϕ does not belong to the core of Φ. *Examples*: 'pq' does not belong to the core of '$pq \vee qs \vee q\bar{r} \vee \bar{q}r\bar{s}$', for '$s \vee \bar{r} \vee r\bar{s}$' is tautologous; whereas 'qs' belongs to the core, for '$p \vee \bar{r}$' is not tautologous.

That the criterion is geared only to irredundant formulas is no limitation, for we have seen how to make a formula irredundant. That the criterion enables us to spot as core clauses only clauses already present is again no limitation, for, by definition, in any irredundant formula all the core clauses are present. Actually uniliteral irredundancy would suffice here without clausal; but it is more efficient to start from a fully irredundant formula, since there are then fewer clauses to examine.

It remains to justify the criterion, by proving that a clause of an irredundant formula belongs to the core if and only if it meets the stated test. In other words, what is wanted is the theorem:

(IX) *A clause ϕ of an irredundant Φ belongs to the core of Φ if and only if there is an assignment of truth values to letters not in ϕ that falsifies all clauses of Φ, other than ϕ, that oppose ϕ in at most one letter.* (I assume as usual that Φ is not simply a tautology $\alpha \vee \bar{\alpha}$.)

*Proof.** Let Φ be any irredundant formula, ϕ any clause of Φ, and Ψ the alternation of all prime implicants of Φ but ϕ. Then ϕ is in the core of Φ if and only if Ψ is not equivalent to $\phi \vee \Psi$ (and thus to Φ); hence if and only if ϕ does not imply Ψ. So what is to be proved is (a) that if ϕ does not imply Ψ then some assignment to letters not in ϕ falsifies all clauses of Φ, other than ϕ, that oppose ϕ in at most one letter, and (b) that if ϕ implies Ψ then each assignment to letters not in ϕ is compatible with some clause of Φ, besides ϕ, that opposes ϕ in at most one letter.

Proof of (a). Since ϕ does not imply Ψ, some assignment A to the letters of Φ verifies ϕ and falsifies Ψ. Let A' be the part of A that has to do with letters not in ϕ, and let ψ be any clause of Φ other than ϕ that opposes ϕ in at most one letter; to show that A' falsifies ψ. *Case* 1. ψ opposes ϕ in no letter. Then, since A verifies ϕ, A falsifies no literals of ψ whose letters are in ϕ; but A does falsify ψ, for ψ is clearly a clause of Ψ, and A falsifies Ψ; so A' falsifies ψ. *Case* 2. ψ has just one literal, call it ξ, opposed to a literal of ϕ. But ψ and ϕ are not simply ξ and ζ, or Φ would be a tautology. So ψ and ϕ have a consensus, and it, since it implies $\psi \vee \phi$ and therefore Φ, subsumes a prime implicant χ of Φ. Since χ lacks ζ, χ is not ϕ; hence χ is a clause of Ψ; and hence A falsifies χ. But all literals of χ whose letters are in ϕ are identical with literals of ϕ, and hence verified by A; so A' falsifies at least one literal of χ. But any such literal of χ is a literal of ψ. So A' falsifies ψ.

Proof of (b). Here the assumption is that ϕ implies Ψ; *i.e.*, that Φ and Ψ are equivalent. Let A' be any assignment to the letters not in ϕ. Let Φ' and Ψ' be what Φ and Ψ reduce to under A'; thus Φ' and Ψ' are equivalent. It will be sufficient to show that Φ' has a clause, other than ϕ, that opposes ϕ in at most one letter. Now ϕ, being a clause still of Φ', implies Φ' and therefore Ψ'. Hence ϕ must subsume some clause ψ' of Ψ'; for ϕ subsumes every clause of Ψ' not opposed to ϕ, there being no further letters. This ψ' is the residue in Ψ' of some clause ψ of Ψ. Since ϕ is not a clause of Ψ, ψ is not ϕ; moreover, since any clause of Ψ is a prime implicant of Φ, ψ does not even subsume ϕ. So ψ lacks a literal ζ of ϕ; ϕ subsumes $\zeta\psi'$. Now assign falsity to ζ and truth to the rest of the literals of ϕ. This assignment verifies ψ', and therewith Ψ'; hence also Φ', which is equivalent to Ψ'. Therefore this assignment verifies some clause of Φ'. But a clause of Φ', to be thus verified, must consist solely of literals of ϕ other than ζ, plus perhaps ξ; hence it is other than ϕ and opposes ϕ in at most one letter.

* Previously proved in one of my lectures on "Simplifying truth functions," College of Engineering, University of Michigan, June, 1958.

THE VOTING PROBLEM*

RICHARD STEARNS, Princeton University

1. A pair of binary relations, P and I, are said to *define a pattern S on the elements of a set A* if and only if the three following conditions are satisfied: i) P is antireflexive and antisymmetric, ii) I is reflexive and symmetric, iii) for all a and b in A, aIb if and only if neither aPb or bPa. When aIb, one says that a is *indifferent* to b and when aPb, one says that a is *preferred over b*.

A pattern is *strong* if aIb implies $a=b$ and is *ordered* if both P and I are transitive.

A set of strong ordered patterns defined on the elements of a finite set A by various pairs of binary relations is called a *set of voters* and *generates* a new pair of relations on the set according to the following rule: aPb if and only if a is preferred over b in a majority of the voters and aIb if and only if neither aPb or bPa. It is obvious that the relations P and I so defined satisfy conditions i, ii, and iii above and thus P and I define a pattern S. This pattern S is called the pattern *generated* by the set of voters.

It was shown in a paper by McGarvey† that a set of voters can always be found to generate an arbitrary pattern. This was done with a construction which required $n(n-1)$ voters for a pattern of n elements. The voting problem considered here is that of finding the smallest number of voters required to generate an arbitrary pattern of n elements. First, it will be shown that this number is no larger than $n+1$ when n is odd or $n+2$ when n is even.

2. Using the notation of McGarvey, a strong ordered pattern S defined on the elements of $A=\{a_i: i=1, \cdots, n\}$ will be represented by the following: $S=a_1, \cdots, a_n$ where a_iPa_j if and only if a_i is to the left of a_j.

If $S_i=a_{i,1}, \cdots, a_{i,r_1}$ and all the S_i are disjoint, then define $S_1, \cdots, S_m=a_{1,1}, \cdots, a_{1,r_1}, \cdots, a_{m,1}, \cdots, a_{m,r_m}$.

3. Suppose S is an arbitrary pattern on $2k$ elements, a_i $(i=1, \cdots, 2k)$. Define for $1 \leqq i \leqq k$:

$$A_{1,i}=\{a_j: a_jPa_{2i-1} \text{ and } a_jPa_{2i}\},$$
$$A_{2,i}=\{a_j: \sim(a_jPa_{2i-1}), a_jPa_{2i}, \text{ and } j \neq 2i-1\},$$
$$A_{3,i}=\{a_j: \sim(a_jPa_{2i-1}), \sim(a_jPa_{2i}), \text{ and } 2i-1 \neq j \neq 2i\},$$
$$A_{4,i}=\{a_j: a_jPa_{2i-1}, \sim(a_jPa_{2i}), \text{ and } j \neq 2i\}.$$

For a fixed i, each of the elements of $A-\{a_{2i-1}, a_{2i}\}$ falls into one and only one

* This paper is the result of some work started by the author as an undergraduate research assistant at Carleton College under a grant from the National Science Foundation to the Mathematics Department.

† David C. McGarvey, A theorem on the construction of voting paradoxes, Econometrica, vol. 21, 1953, pp. 608–610.

of the $A_{j,i}$, depending on its relation to a_{2i-1} and a_{2i}. Also, a_{2i-1} and a_{2i} do not belong to any $A_{j,i}$.

Let $T_{j,i}$ be the strong ordered pattern defined on $A_{j,i}$ by letting a_rPa_s if and only if $r<s$. Let $\overline{T}_{j,i}$ be the pattern on $A_{j,i}$ with the reverse order.

Furthermore, define $Q_i=a_{2i-1}, a_{2i}$ if $a_{2i-1}Pa_{2i}$ in S; otherwise let $Q_i=a_{2i}, a_{2i-1}$. Finally, let $R_i=Q_i$ if $\sim(a_{2i-1}Ia_{2i})$; but if $a_{2i-1}Ia_{2i}$, let $R_i=a_{2i-1}, a_{2i}$, the reverse of Q_i.

Now consider the following set of voters:

$$V_{2i-1} = T_{1,i}, a_{2i-1}, T_{2,i}, a_{2i}, T_{3,i}, T_{4,i} \qquad (1 \leqq i \leqq k),$$
$$V_{2i} = a_{2i}, \overline{T}_{4,i}, a_{2i-1}, \overline{T}_{3,i}, \overline{T}_{2,i}, \overline{T}_{1,i} \qquad (1 \leqq i \leqq k),$$
$$V_{2k+1} = Q_k, \cdots, Q_1, \qquad V_{2k+2} = R_1, \cdots, R_k.$$

A routine inspection of this set of voters shows that it generates the desired pattern S. For example, consider what happens when $a_{2r}Pa_{2s}$ in S. For $r \neq i \neq s$, the effects of V_{2i} and V_{2i-1} cancel each other with respect to these two elements. The crucial voters in this case are seen to be V_{2r-1}, V_{2r}, V_{2s-1}, and V_{2s}. Since a_{2r} is in either $A_{1,s}$ or $A_{2,s}$ (depending on its relation to a_{2s-1}) and a_{2s} in either $A_{3,r}$ or $A_{4,r}$, three of these four crucial voters (including V_{2r} and V_{2r-1}) will have $a_{2r}Pa_{2s}$ and only one will have the reverse relation. This two-vote difference generates the desired relation. If a_{2r} and a_{2s} were indifferent, V_{2r} and V_{2r-1} would cancel V_{2s} and V_{2s-1} and the vote would be a tie. The proper relationship between a_{2r-1} and a_{2r} is established by the last two voters because of the way Q_r and R_r are ordered. Other pairs of elements check in a similar manner. *S has been generated with $2k+2$ voters.*

4. Now assume that S has $2k+1$ elements. If possible, choose an element to be called a_{2k+1} such that either (Case 1) $B=\{x: xPa_{2k+1}\}$ has an even number, $2r$, of elements or (Case 2) B has $2r+1$ elements and there is a $b\in B$ and a $c\in(A-B-\{a_{2k+1}\})$ such that $\sim(cPb)$. In case one, name the other elements so that $a_i \in B$ $(i=1, \cdots, 2r)$. In case two, name them so that $a_i \in B$ $(i=1, \cdots, 2r+1)$, $b=a_{2r+1}$, and $c=a_{2r+2}$.

Now define:

$$V_{2i-1} = T_{1,i}, a_{2i-1}, T_{2,i}, a_{2i}, T_{3,i}, T_{4,i} \qquad (1 \leqq i \leqq k),$$
$$V_{2i} = a_{2i}, \overline{T}_{4,i}, a_{2i-1}, \overline{T}_{3,i}, \overline{T}_{2,i}, \overline{T}_{1,i} \qquad (1 \leqq i \leqq k),$$
$$V_{2k+1} = a_{2k+1}, Q_k, \cdots, Q_1,$$
$$V_{2k+2} = R_1, \cdots, R_r, a_{2k+1}, R_{r+1}, \cdots, R_k \text{ (Case 1)}$$
$$V_{2k+2} = R_1, \cdots, R_r, a_{2r+1}, a_{2k+1}, a_{2r+2}, R_{r+2}, \cdots, R_k. \text{ (Case 2)}$$

A check of the various pairs of elements shows that this set generates S. Without a_{2k+1}, these voters are the same as those for $2k$ elements. The proper relation between a_{2k+1} and a_{2i} or a_{2i+1} is established by V_{2i-1}, V_{2i}, V_{2k+1}, and V_{2k+2}, the other pairs canceling.

Now assume that no point of A can be found for a_{2k+1} that satisfies either of

the two cases. *Then for all x, y, and z in A, either xPy, yPz, and zPx in S or the reverse relations hold.*

Proof. Assume that xIy in S for some distinct x and y. If Case 1 cannot be applied, there is an odd number of elements preferred over x and hence some z such that zPx. If yPz, apply Case 2 by letting $x=c$, $y=b$, and $z=a_{2k+1}$. Otherwise, apply Case 2 by letting $x=a_{2k+1}$, $y=c$, and $z=b$.

Now assume that S is strong. If xPy, yPz, and xPz for some x, y, and z and Case 1 does not apply, Case 2 applies by letting $x=b$, $y=a_{2k+1}$, and $z=c$. Thus the theorem holds.

A check of all the possible ways four elements can be related in a strong pattern reveals that the condition of the above theorem cannot hold for patterns on four or more elements. Thus the only pattern which does not fit into one of the two cases is the three element pattern on $\{a, b, c\}$ with aPb, bPc, and cPa. Since this can be generated with three voters $(a, b, c; c, a, b;$ and $b, c, a)$, *the arbitrary pattern on $2k+1$ elements has been generated with $2k+2$ voters or less.*

5. How good are these results? It would not be surprising if fewer voters are required for large n because large subsets of the elements in the pairs of voters have been ordered merely to cancel each other rather than to contribute constructively to the generation of the pattern. Nevertheless, the result is known to be exact for $n=3$, 4, and 5. It will now be shown that, *for large n, the number of voters required is greater than $.55n/(\log n)$.*

Proof. The number of patterns with n elements is $3^{\binom{n}{2}}$, there being three possible relations between each pair of elements. The number of patterns that can be generated with v voters on n elements is surely less than $(n!)^v$, the number of ordered sets of voters.

Any pattern that can be generated by exactly k voters can also be generated with exactly $k+2$ voters by adding two voters which cancel each other. If all the patterns on n elements can be generated with v or less voters (and consequently with exactly v or $v-1$ voters), the number of possible patterns that can be generated with exactly v or $v-1$ voters cannot be smaller than the number of patterns. Thus: $3^{\binom{n}{2}} \leqq (n!)^{v-1}+(n!)^v<2(n!)^v$.

Now it is well known that $n!=pn^ne^{-n}(2\pi n)^{1/2}$ where p goes to one as n gets large. Substituting into the inequality and taking logarithms:

$$\frac{n(n-1)}{2}\log 3 < \log 2 + v[\log p + n\log n - n + \tfrac{1}{2}\log n + \tfrac{1}{2}\log 2\pi],$$

$$\frac{n-1}{2}\log 3 < \frac{\log 2}{n} + v\left[\frac{\log p}{n} + \log n - 1 + \frac{\log n}{2n} + \frac{\log 2\pi}{2n}\right].$$

For large n, $\frac{1}{2}(n-1)\log 3<v(\log n-1)$ or $v>.55n/(\log n)$.

6. Thus, for large n, the number of voters required to generate an arbitrary pattern lies between $n+2$ and $.55n/(\log n)$.

THE RELATION BETWEEN A COMPACT LINEAR OPERATOR AND ITS CONJUGATE

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1. Introduction. We present a systematic account of known theorems relating compact linear operators and their conjugates. Examples are given showing that all the theorems which are "possible" in a certain broad sense are already known. The general method is that of [6].

In what follows, X and Y are normed linear spaces. $[X, Y]$ is the set of bounded linear operators with domain X and range in Y. T denotes a linear operator and $R(T)$ is its range. T is compact if, for each bounded sequence (x_n) in X, (Tx_n) has a convergent subsequence. $[X, Y]_c$ stands for the set of compact linear operators with domain X and range in Y. We say that T has an inverse if $Tx=0$ implies $x=0$, *i.e.*, if T sets up a 1-1 mapping of X onto $R(T)$. The inverse mapping T^{-1} is also linear. X' is the space of bounded linear functionals on X, normed in the usual way. If $T\in[X, Y]$, the operator T' is defined as follows: $T'y'=x'$, where $x'\in X'$ is defined by $x'(x)=y'(Tx)$, all $x\in X$. $T'\in[Y', X']$ and $\|T'\|=\|T\|$.

Motivated by the known theorems relating a bounded operator T and its conjugate, we classify various possibilities for T by:

I: $R(T)=Y$ (indicated by writing $T\in I$).

II: $R(T)\neq Y$ but $\overline{R(T)}=Y$ (written $T\in$ II, and so on for succeeding cases).

III: $\overline{R(T)}\neq Y$.

1: T^{-1} exists and is bounded.

2: T^{-1} exists but is not bounded.

3: T^{-1} does not exist.

If $T\in$ II and $T\in 1$, we combine this by writing $T\in \mathrm{II}_1$. Thus there are nine possibilities for T. Similarly, T' has nine classifications. Thus, the pair (T, T') has 81 classifications. We call these 81 classifications the states of the pair (T, T'). If, for example, $T\in \mathrm{II}_1$ and $T'\in \mathrm{III}_3$, we say that the pair is in state $(\mathrm{II}_1, \mathrm{III}_3)$.

Taylor and Halberg [6] found the possible states for the pair (T, T') when $T\in[X, Y]$. They organized their results schematically as shown in Figure 1. Referring to this figure, if a box is crossed out, this means the corresponding state is impossible for any pair (T, T'), regardless of the choice of X and Y. If, on the other hand, a box is not crossed out, it is "possible." Some "possible" states become impossible when X and Y are suitably restricted. This is symbolically noted in the square representing the state. The key below the figure gives the meanings of the symbols.

Remark. A generalization to unbounded operators of the Taylor-Halberg

state diagram for bounded operators has been made by Goldberg [4]. The details are to appear in the Pacific Journal of Mathematics and the results are:

1) If $T: X \to Y$ is an operator whose graph is closed in the product topology of $X \times Y$ and the domain of T is dense in X, then the Taylor-Halberg state diagram remains true. (It is remarkable that such a vast extension of the operator class does not result in the opening up of a single new square in the state diagram.)

2) If $T: X \to Y$ is *any* linear operator with dense domain (Linear is being used to mean $T(ax+by)=aTx+bTy$; there are no topological implications.), the state diagram is the same as the Taylor-Halberg state diagram except that all the X and Y_r symbols are to be erased.

State Diagram for Bounded Linear Operators ($[X, Y]$)

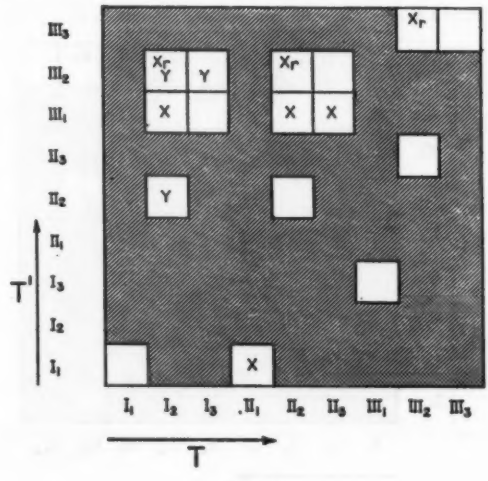

FIG. 1

Key: X: cannot occur if X is complete; Y: cannot occur if Y is complete; X_r: cannot occur if X is reflexive.

2. Derivation of the state diagram for compact operators. An operator is bounded if and only if it sends bounded sequences into bounded sequences. Convergent sequences are bounded. Therefore every compact operator is a bounded operator so $[X, Y]_c$ is a subset of $[X, Y]$. Hence the state diagram for $[X, Y]_c$ can be thought of as a restriction (fewer open squares) of the diagram for $[X, Y]$. The restriction of the state diagram for $[X, Y]$ to the state diagram for $[X, Y]_c$ follows from two lemmas below. Because of the form of Lemma 1, the state diagram will be established under the assumption that X and Y are

infinite dimensional. The simple case in which X and/or Y is finite dimensional is considered separately.

LEMMA 1. *If T is compact and $T \in \mathbf{1}$, then* dim $R(T)$ *and* dim X *are finite and equal* ([5], p. 115).

From Lemma 1, if X is infinite dimensional and T is compact, state 1 is impossible for T. This means that the first, fourth, and seventh columns of the $[X, Y]$ state diagram correspond to impossible states in the $[X, Y]_c$ state diagram. Similarly, Y' is infinite dimensional as a consequence of our assumption that Y is infinite dimensional, so the first, fourth and seventh rows are impossible states in the $[X, Y]_c$ state diagram. Thus we see from Lemma 1 that seven of the sixteen possible states for $[X, Y]$ are impossible for $[X, Y]_c$. To see the full strength of Lemma 1, observe that forty-five squares, more than half the total, are shown to be impossible by Lemma 1, without invoking any other theorems.

State Diagram for Compact Linear Operators ($[X, Y]_c$)

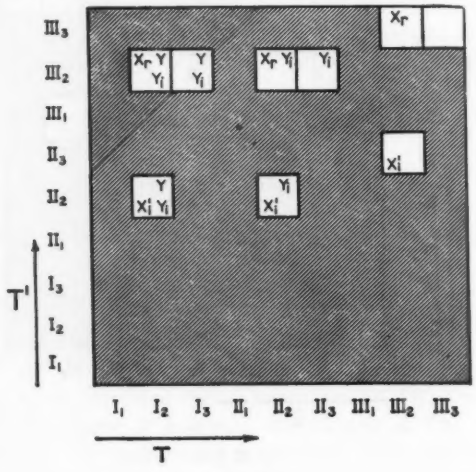

FIG. 2

Key: Y: impossible if Y is complete; X_r: impossible if X is reflexive; $Y_i(X_i')$: impossible if $Y(X')$ is inseparable.

The following result of Banach shows that the state diagram for compact operators is further restricted for certain choices of X' or Y.

LEMMA 2. *If T is compact, then $R(T)$ is separable* ([1], p. 96).

If T is not in III and Y is inseparable, then $\overline{R(T)}$ is inseparable. Hence $R(T)$ is inseparable and Lemma 2 shows that T is not compact. Thus, if T is compact, states I and II are impossible and the first six columns are deleted.

Similarly, if X' is inseparable the first six rows are impossible states. The resultant state diagram is shown in Figure 2. The examples of Section 3 show that this is the final form.

Remark. In constructing the state diagram for X and/or Y finite dimensional, it is simplest to consider in turn each of the three cases listed below. The results given are immediate. If neither X nor Y is (0), the two states shown in each case always exist. No others exist.

1. dim X = dim Y; X and Y are both finite dimensional. (III_3, III_3), (I_1, I_1).
2. dim X is greater than dim Y; Y is finite dimensional. (III_3, III_3), (I_3, III_1).
3. dim Y is greater than dim X; X is finite dimensional. (III_3, III_3), (III_1, I_3).

Obviously II and 2 are impossible for T and T' so the state diagram (Fig. 3) is drawn with only the 16 squares listed.

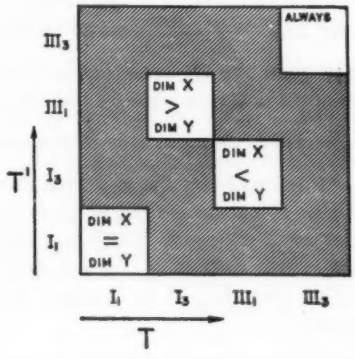

FIG. 3

Key: The four open squares correspond to existing states if and only if the conditions within the open squares are fulfilled.

3. All the states shown as possible exist "maximally." The examples to follow show that every state shown as "possible" actually exists. The examples are "maximal" in the sense that (1) X and Y are each complete, or even reflexive, and (2) X' and Y are inseparable, unless this is already forbidden by the $[X, Y]_e$ state diagram. Making X complete and Y reflexive is suggested by the occurrence of these restrictions in the $[X, Y]$ state diagram, for they do not appear as restrictions in the $[X, Y]_e$ state diagram.

(III_3, III_3): All the operators with finite-dimensional ranges are in this state for any infinite-dimensional (X, Y) pair. An (X, Y) pair such that X and Y are reflexive, and X' and Y are inseparable, is: $X = Y = l^2(Q)$ where the cardinality of Q is greater than $\aleph_0$. Note: For any set Q, $l^2(Q)$ is defined as the set of those

scalar-valued functions with domain Q such that (1) at most a countable number of the coordinates are nonzero and (2), $\sum|x_q|^2$ is finite. x_q is the qth coordinate of a typical function x. The norm of x is the square root of the sum in (2).

(II_2, II_2): $X=Y=l^2$; let (u_k), $k=1, 2, \cdots$ be a countable orthonormal basis in l^2. Define T by $Tu_k=2^{1-k}u_k$. T is compact because it satisfies the criterion (see, e.g., [3], Th. 7, Cor.) $\sum|t_{ij}|^2$ is finite, where (t_{ij}) is the matrix corresponding to T. $T'=T$; hence the state must be, according to the diagram, (II_2, II_2) or (III_2, III_2). But $\overline{R(T)}=Y$ because every element with at most a finite number of nonzero coordinates is in $R(T)$.

(II_2, III_2): $X=Y=l^2$; define T by $Tu_k=2^{1-k}u_{k-1}$, $k=2, 3, \cdots$ and $Tu_1=0$. The arguments of the preceding example can be used to show T is compact and in (II_2, III_2). To make this example "maximal," we must modify it so that X is inseparable. Let X_0 be the direct sum of l^2 and $l^2(Q)$, where $l^2(Q)$ is a nonseparable Hilbert space. Define T_0 as the direct sum of T_1 and T by setting $T_1=0$ on $l^2(Q)$. This T_0, with the pair (X_0, Y), is the desired example.

This device can be used to make X' or Y inseparable in all the following state examples, unless the state diagram already forbids this.

(III_2, II_2): Use the conjugate of the operator in the preceding example.

(II_2, III_2): $X=l^1$, $Y=l^2$; define T by $Tu_k=2^{1-k}u_k$, as in example (II_2, II_2). It is shown in [6], p. 104 that the state is (II_2, III_2). Compactness is shown as follows: Let T_0 be the same as T except that the domain of T_0 is l^2. Let I_0 be the canonical imbedding of l^1 in l^2 defined by $I_0x=x$. It is readily verified that $\|I_0\|=1$. Notice that $T=T_0I_0$ and that T_0 is compact by the previously used criterion. Hence T is compact.

(III_2, III_2): $X=l^1$, $Y=l^2$; $Tu_k=2^{-k}u_{k+1}$, $k=1, 2, \cdots$; T is shown to be compact by the method of the previous example.

(I_2, II_2), (I_2, III_2), (I_3, III_2): Examples of these states are obtained by modifying the previously given examples of the states (II_2, II_2), (II_2, III_2), (II_3, III_2), respectively, using the procedure given in [6].

4. The state diagram for weakly compact operators is the same as that for bounded operators. The weak topology on X is the weakest (coarsest, smallest) topology making every element of X' continuous. A linear operator $T\in[X, Y]$ is weakly compact if the weak closure of $T(S_X)$ is compact in the weak topology on Y (S_X is the unit sphere in X). The set of weakly compact operators is designated by $[X, Y]_{wc}$. Observe that $[X, Y]_c$ is a subset of $[X, Y]_{wc}$ and that $[X, Y]_{wc}$ is itself a subset of $[X, Y]$. To show the second inclusion, note that a weakly compact set is weakly bounded hence, by the uniform boundedness principle it is norm-bounded. The first inclusion is true because $T\in[X. Y]_c$ means that the norm closure of $T(S_X)$ is norm-compact and hence weakly com-

pact. Now note that the norm closure of a manifold equals the weak closure, therefore the weak closure of $T(S_X)$ is weakly compact.

We now show the equality of state diagrams for $[X, Y]$ and $[X, Y]_{wc}$. We need the following result:

THEOREM. *Y is norm reflexive if and only if S_Y is weakly compact* (see, *e.g.*, [2], Ch. IV, Sec. 5, n° 2, Prop. 6).

This theorem shows us that $[X, Y] = [X, Y]_{wc}$ if Y is reflexive. Thus the examples in [6] which show a state is possible and have Y reflexive also show the state is possible for $[X, Y]_{wc}$. The only possible states in the $[X, Y]$ diagram not included by this are (I_2, II_2), (I_2, III_2), (I_3, III_2). But if T is compact, T is weakly compact so the examples of these three states given in the $[X, Y]_c$ diagram suffice for the $[X, Y]_{wc}$ diagram.

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A CONSTRUCTION OF THE RATIONAL NUMBERS

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1. Introduction. Constructing the real numbers from the rationals is necessary for an understanding of what the real numbers are. Not so the construction of the rationals starting from the natural numbers. Yet the possibility of such a construction is an interesting, classical, and elementary fact. Knowing this fact by having seen the construction is worthwhile, provided that the cost is not too great: the proof should either be quick and easy, as Landau estimated it to be ([1], Preface for the Teacher), or it should at the same time teach something else of value, as Thurston has made it do [2]. The new method given here is probably not easier than the well-known ones, but it uses algebraic methods which hopefully lend it new interest. Such a treatment would go well in an abstract algebra course, if not in beginning analysis.

The usual motivation for signed integers is arithmetical: observing that sub-

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traction of natural numbers is only sometimes possible, one contrives artificial new numbers to serve as differences of natural numbers. The point of view we have adopted is rather geometrical: the integers arise as a group of transformations. My intuitive preference for this point of view, apparently also preferred by Alfred North Whitehead [3], derives perhaps from the directness with which most of the algebraic properties of addition fall out. Indeed, if we use functions from the natural numbers to the natural numbers to represent integers, and functions from the integers to the integers to represent rationals, the following things fall immediately into place: the operations of addition and multiplication, with their associative laws, identities and inverses, come spontaneously into being as instances of composition of functions; the usual recursive definition of addition is seen to be the assertion that the functions being "added" are endomorphisms of the Peano succession of natural numbers; the left distributive law, of which the recursive definition of multiplication is a special case, is the assertion that the functions being "multiplied" are endomorphisms of the additive group; the absence of a multiplicative inverse for 0 is the inevitable result of the function representing 0 not being one to one. Note also how the principle of mathematical induction becomes one instance of a more general phenomenon, the use of a generator for an algebraic system.

Assuming some knowledge of sets, functions and groups, as we do, is not only for the sake of brevity; without prior knowledge of these, this would be hard reading. What is, of course, studiously avoided is any reference to a previous knowledge of arithmetic.

2. Successions. A pair (N, s), where N is a set and s a function mapping N into N, we shall call a *succession*. Subsuccessions are formed by restricting s to subsets of N which are closed under s. Such subsets we shall call *tails*: A is a tail when $s(A) \subset A$. We shall loosely write (A, s) for the resulting subsuccession. We call (N, s) a *Peano* succession when the following three axioms are satisfied:

INVERSION AXIOM. *s is one to one (s^{-1} is a function).*

INFINITY AXIOM. $s(N) \neq N$.

INDUCTION AXIOM. (N, s) *has a* generator, *that is, an element e of N such that N is the only tail containing e.*

A *homomorphism* of a succession (N_1, s_1) into a succession (N_2, s_2) is a function f mapping N_1 into N_2 such that $f(s_1(x)) = s_2(f(x))$ for every x in N_1. A homomorphism is an *isomorphism* if it is one to one and onto. All Peano successions are isomorphic, and they are regarded as models for the natural number system. We omit the proof of this uniqueness theorem, along with many other important things.

2.1. *If e is a generator for a succession (N, s), then $s(N) \cup \{e\} = N$.*

Proof. The set $s(N) \cup \{e\}$ is a tail containing e.

2.2. *If (N, s) is a Peano succession with generator e, then $s(N)=N-\{e\}$, and $(s(N), s)$ is a Peano succession with generator $s(e)$.*

Proof. Since $s(N)$ is a tail and $s(N)\neq N$, $e\notin s(N)$ by the induction axiom. Using this and 2.1, $s(N)=N-\{e\}$. If A is a tail contained in $s(N)$ and containing $s(e)$, then $A\cup\{e\}$ is a tail in N containing e. Hence $A\cup\{e\}=N$ and $A=s(N)$. Thus $s(e)$ is a generator for $(s(N), s)$, and the other axioms for $(s(N), s)$ are trivial.

3. Cyclic groups. We shall write groups additively. A group G is called *cyclic* if it has a *generator*, that is, an element 1 of G such that G is the only subgroup containing 1. (Compare this with the Induction Axiom for successions.)

3.1. *Every cyclic group is Abelian.*

Proof. The *centralizer* C_x of an element x of G is the set of all y such that $x+y=y+x$. It is easy to see that C_x is a subgroup, and that $x\in C_x$. In particular, for any generator 1 of G, $C_1=G$. Now the *center* C of G is by definition the intersection of all C_x, therefore a subgroup. Also $x\in C$ if and only if $C_x=G$. Having shown that $C_1=G$, we have $1\in C$, and therefore $C=G$. Thus G is Abelian.

We probe the structure of G by looking at the successions generated by its elements. For any y, the *translation* by y is the function T_y defined by $T_y(x)=y+x$ for all x. Note that T_0 is the identity function. Let 1 be a generator. In the succession (G, T_1) the intersection of all tails containing 0 is a tail, the smallest tail containing 0, called the *ray* through 1, and denoted G_1. By its definition the succession (G_1, T_1) satisfies the inversion and induction axioms. If for a generator 1 (G_1, T_1) is actually a Peano succession, we say that G is *free*. All free cyclic groups are isomorphic (as will be seen from Theorem 2), and they are models for the group of the integers. We write $-G_1$ for $\{x: -x\in G_1\}$.

3.2. *If 1 is a generator, then $G=G_1\cup(-G_1)$.*

Proof. Let $A=G_1\cup(-G_1)$, and let $H=\{x: T_x(A)\subset A\}$. Then H is a subgroup. For if $T_x(A)\subset A$ and $T_y(A)\subset A$, then $T_{x+y}(A)=T_x(T_y(A))\subset A$, and $T_{-x}(A)=-T_x(-A)\subset A$, because $-A=A$. Furthermore it is easy to show using 2.1 that $T_1(A)\subset A$, so that $1\in H$. Thus $H=G$. This shows that A is closed under addition. Since $A=-A$, A is a subgroup, and since $1\in A$, $A=G$.

3.3. *G_1 is closed under addition.*

Proof. We use induction in (G_1, T_1). The set M of those x in G_1 such that $T_x(G_1)\subset G_1$ is a tail. For if $x\in M$, then $T_{1+x}(G_1)=T_1(T_x(G_1))\subset T_1(G_1)\subset G_1$. Since obviously $0\in M$, $M=G_1$.

3.4. *If G is a free cyclic group with 1 a generator, then $G_1\cap(-G_1)=\{0\}$.*

Proof. Let $A=G_1-\{0\}$. We must show that A and $-A$ are disjoint. If

$x \in A \cap (-A)$, then x and $-x$ belong to A. By 2.1 $x = 1+y$ for some y in G_1. Then $y+(-x) \in G_1$ by 3.3, and $1+y+(-x) = x+(-x) = 0$. Thus $0 \in T_1(G_1)$, which according to 2.2 is impossible, since by hypothesis (G_1, T_1) is a Peano succession.

The partition of G into two closed-under-addition sets like this is tantamount to the introduction of an order in G, making G into a linearly ordered group: one arbitrarily chooses one of the generators, 1 or -1, to be the *positive* one (say 1 is chosen) and defines $x \leqq y$ to mean $y+(-x) \in G_1$. We shall ignore this possibility, but we shall need these lemmas in Section 6.

4. Skew-fields. A *skew-field* is a set F having more than one element, with two operations, addition and multiplication, such that F with addition is an Abelian group, multiplication is an associative operation with an identity 1 and inverses for all elements except 0 (the additive identity), and both the distributive laws hold: $(x+y)\cdot z = x\cdot z + y\cdot z$ and $z\cdot(x+y) = z\cdot x + z\cdot y$. A skew-field is *prime* if it has no proper subfields. It is a *field* if multiplication is commutative.

4.1. *Every prime skew-field is a field.*

Proof. The set C_x of all y such that $x\cdot y = y\cdot x$ is closed under all the operations of F. Since $1 \in C_x$, C_x is a subfield, whence $C_x = F$.

We probe the structure of a prime field F by looking at the subgroups generated by its elements. For any x in F, the intersection of all (additive) subgroups containing x is a subgroup F_x called the subgroup through x. In particular F_1 is called the *integer group* of F. The integer group is automatically cyclic; if it is free, we say that F has *characteristic zero*. All prime fields of characteristic zero are isomorphic, as will be seen from Theorem 4, and they are models for the rational number system.

5. From a Peano succession to a free cyclic group.

THEOREM 1. *Given a succession (N, s) satisfying the inversion and induction axioms, there exists a cyclic group I such that (N, s) is isomorphic to the ray (I_1, T_1) through a generator of I; I is free if and only if (N, s) is a Peano succession.*

The proof is carried out by the numbered steps which follow. It can be outlined thus. In 5.1 we show that (N, s) can be isomorphically represented by a succession consisting of its endomorphisms. Composition of endomorphisms gives us an addition, but the result may not be a group because the inverse of an endomorphism (which exists by 5.2) may not be an endomorphism, its domain not being all of N. Now look at the end product: I is isomorphically represented by its translations, and these act in I_1 (the image of N) as homomorphisms of tails. This suggests defining the elements of I to be homomorphisms of tails in N, which we do, except that homomorphisms which "agree" are identified.

In this section we denote composition of functions with values in N by the addition sign: $(f+g)(x)=f(g(x))$ for all x for which this makes sense. An *endomorphism* of (N, s) means a homomorphism of (N, s) into itself.

5.1. *To each x in N there corresponds a unique endomorphism f_x such that $f_x(e)=x$. Every endomorphism is one of these, f_e is the identity function on N, and $f_{s(x)}=s+f_x$ for all x in N.*

Proof. Induction on x: Let M be the set of those x such that such an endomorphism exists. Then $e\in M$ because for f_e you can use the identity, and M is a tail, because if f_x is already in hand, for $f_{s(x)}$ you can use $s+f_x$. Hence $M=N$. To prove the uniqueness, suppose f and g are endomorphisms such that $f(e)=g(e)=x$; then the set where f and g agree is a tail containing e, hence all of N, and $f=g$. If f is any endomorphism whatever, let $x=f(e)$; then by the uniqueness just proved $f=f_x$.

5.2. *All endomorphisms of (N, s) are one to one.*

Proof. Induction on x: Let M be the set of those x such that f_x is one to one. Then $e\in M$ because the identity is one to one, and M is a tail because, s being one to one, if f_x is one to one so is $f_{s(x)}=s+f_x$. Therefore $M=N$.

5.3. *If A is a tail and f an endomorphism, then $f(A)\subset A$.*

Proof. Induction on x: For any A let M be the set of those x such that $f_x(A)\subset A$. Then $e\in M$ because $f_e(A)=A$, and M is a tail because $s(A)\subset A$ so that if $f_x(A)\subset A$ then $f_{s(x)}(A)=s(f_x(A))\subset s(A)\subset A$. Thus $M=N$.

5.4. *If A and B are nonempty tails, then $A\cap B$ is a nonempty tail.*

Proof. $A\cap B$ is obviously a tail. Now let $x\in A$ and $y\in B$. Then by 5.3, $f_x(y)\in B$. Now the set of those z such that $f_z(z)\in A$ is a tail containing e ($f_x(e)=x\in A$), so all of N. In particular $f_x(y)\in A$, and $A\cap B$ is not empty.

Define a *translation* of (N, s) as a homomorphism of a nonempty tail of (N, s) into (N, s). Endomorphisms are translations, namely those with domain N. Addition of translations is composition, in accordance with our announced notational policy. The importance of 5.4 is to insure that a sum of translations shall have a nonempty domain. Indeed, if f and g are translations with domains D_f and D_g, then $g(D_g)$, the range of g, is a nonempty tail which by 5.4 intersects D_f. Hence $D_g\cap g^{-1}(D_f)$, which is the domain of $f+g$, is not empty.

Equality at infinity is a relation between translations, written $f=(\infty)g$, defined thus: $f=(\infty)g$ if and only if $f(x)=g(x)$ for all x in some nonempty tail. Since $\{x: f(x)=g(x)\}$ is a tail in any case, the existence of such an x is sufficient for $f=(\infty)g$ to hold.

5.5. *Equality at infinity is an equivalence.*

Proof. For any translation f, $f=(\infty)f$ because the domain of f is not empty. If

$f=(\infty)g$, then $g=(\infty)f$ trivially. If $f=(\infty)g$ and $g=(\infty)h$, then $f(x)=g(x)=h(x)$ for any x in the intersection of the nonempty tails $\{x: f(x)=g(x)\}$ and $\{x: g(x)=h(x)\}$. Such an x exists by 5.4.

5.6. *If $f_1=(\infty)f_2$ and $g_1=(\infty)g_2$, then $f_1+g_1=(\infty)f_2+g_2$.*

Proof. By repeated use of 5.4 there exists x in the intersection of the nonempty tails $\{x: g_1(x)=g_2(x)\}$, $\{x: f_1(g_1(x))=f_2(g_1(x))\}$, and the domains of f_1+g_1 and f_2+g_2. For such an x, $f_1(g_1(x))=f_2(g_1(x))=f_2(g_2(x))$.

5.7. *For any translation f there are endomorphisms g and h such that $f=(\infty)g+h^{-1}$.*

Proof. Let x belong to the domain of f, let $g=f_{f(x)}$ and let $h=f_x$.

Let the elements of I be the equivalence classes of translations which are equal at infinity. For any translation f let $i(f)$ be the equivalence class to which it belongs. Define addition in I by $i(f)+i(g)=i(f+g)$; this is possible by 5.6. Let $0=i(f_e)$ and $1=i(s)$.

5.8. *I is a cyclic group with 1 for a generator.*

Proof. Addition is associative for translations because it is composition; it is therefore associative in I. The identity is 0. By 5.7 and 5.2 any element of I contains a one to one translation, say f, whose inverse is therefore also a translation. Then $i(f^{-1})$ is $-i(f)$. Finally, to show that 1 is a generator, let G be a subgroup containing 1. Let $M=\{x: i(f_x)\in G\}$. Then $e\in M$ because $0\in G$, and M is a tail because $i(f_x)\in G$ and $1\in G$ imply $i(f_{s(x)})=i(s+f_x)=1+i(f_x)\in G$. Thus $M=N$ and $i(f)\in G$ for every endomorphism f. By 5.7, $i(f)\in G$ for every translation f, and $G=I$.

5.9. *Define ϕ mapping N into I by $\phi(x)=i(f_x)$. Then ϕ is an isomorphism of (N, s) with (I_1, T_1).*

Proof. We have shown in the proof of 5.8 that ϕ maps N into I_1. Now $\phi(e)=i(f_e)=0$, and for all x, $\phi(s(x))=1+i(f_x)=T_1(\phi(x))$, so that ϕ is a homomorphism, whose range is a tail containing 0, therefore all of I_1. To show that ϕ is one to one we must show that distinct endomorphisms are never equal at infinity. Induction: Let M be the set of x such that if $f(x)=g(x)$ for endomorphisms f and g, then $f=g$. The uniqueness part of 5.1 shows that $e\in M$. If $x\in M$, then $f(s(x))=g(s(x))$ gives $s(f(x))=s(g(x))$, whence $f(x)=g(x)$ and (since $x\in M$) $f=g$. Thus M is a tail, $M=N$, and ϕ is one to one.

We have completed the proof of Theorem 1. That I is free if and only if (N, s) is a Peano system follows immediately from the definition. Note that I is Abelian by 3.1.

The next theorem, while not essential, tidies things up by answering some

uniqueness questions: Does the isomorphism of N with I_1 uniquely determine I? Are the rays through different generators of a cyclic group always isomorphic?

THEOREM 2. *Let G be a cyclic group, 1 a generator of G. Let $N=G_1$ and $s=T_1$. Let I be the group constructed from (N, s) in Theorem 1, and ϕ the isomorphism of (N, s) to (I_1, T_1). Then ϕ can be extended to an isomorphism of G with I.*

Proof. The set of differences $x+(-y)$ with $x, y\in N$ form a subgroup containing $1=1+(-0)$; hence this set is G. Then given any $z=x+(-y)$ we have $z+y=x\in N$, so some element of N goes into N under T_z. Since T_z commutes with T_1, some restriction of T_z is a translation of (N, s), and since all such restrictions are equal at infinity, we can meaningfully define $\psi(z)$ as $i(T_z)$, and ψ maps G into I. It is easy to show that ψ has the required properties of extending ϕ and being an isomorphism.

6. From a free cyclic group to a field.

THEOREM 3. *Given a free cyclic group I, there exists a prime field R of characteristic zero such that I is isomorphic to the integer group of R.*

With one exception, the proof runs parallel to that of Theorem 1, and can be motivated in a similar way. In 6.1 we show that I can be isomorphically represented by a group consisting of its endomorphisms. Composition of endomorphisms gives us a multiplication which in effect makes I into a ring. The isomorphism of the theorem is actually an embedding of this ring in R. Since a field cannot have divisors of 0 (nonzero elements x and y such that $x\cdot y=0$), it is important to know that our ring of endomorphisms has none. This is the content of 6.2. The one place where the argument departs from imitating Theorem 1 is in the proof of 6.2: it is necessary here to use the fact that I is free, as can be seen from the example of the group of integers modulo 6 (for which the conclusion of Theorem 3 is false). Looking now at R, for any y, the *dilation* by y means the function D_y defined by $D_y(x)=y\cdot x$. R is isomorphically represented by its dilations, and these act in R_1 (the image of I) as homomorphisms of subgroups. This suggests defining the elements of R to be homomorphisms of subgroups in I, but again an identification of equivalent homomorphisms must be made.

Note the following change of notation: for functions f and g with values in I, $f+g$ now means the function defined by $(f+g)(x)=f(x)+g(x)$, while composition is indicated as multiplication: $(f\cdot g)(x)=f(g(x))$. As usual 1 is a generator of I.

6.1. *To each x in I there corresponds an endomorphism f_x such that $f_x(1)=x$. Every endomorphism is one of these, f_0 is the constant function mapping I into 0, f_1 is the identity function, and for all x and y, $f_{x+y}=f_x+f_y$, and $f_{-x}=-f_x$.*

Proof. Let H be the set of those x such that such an endomorphism exists. Then $1\in H$ because for f_1 you can use the identity, and H is a subgroup because if f_x and f_y are already in hand, for f_{x+y} and f_{-x} you can use f_x+f_y and $-f_x$ respec-

tively. Since 1 is a generator, $H=G$. To prove uniqueness, suppose f and g are endomorphisms such that $f(1)=g(1)=x$. Then the set where f and g agree is a subgroup containing 1, hence all of G, and $f=g$. If f is any endomorphism, let $x=f(1)$; then by the uniqueness just proved $f=f_x$.

Hereafter when a proof resembles this closely the corresponding proof in Section 5, we shall omit it.

6.2. *All endomorphisms of I except f_0 are one to one.*

Proof. To show that an endomorphism f_x is one to one, we need only show that $f_x(y)\neq 0$ for $y\neq 0$. By 3.2 and symmetry it is sufficient to treat x and y in $A=I_1-\{0\}$. Now A is closed under addition, for if $u, v\in A$, then $u+v\in I_1$ by 3.3, and $u+v$ cannot be 0 since this would require $u=0$ by 3.4, I being free. Let M be the set of those x in A such that $f_x(A)\subset A$. Then M is a tail in (A, T_1), because if $x\in M$ and $y\in A$, then $f_x(y)\in A$, whence $f_{1+x}(y)=y+f_x(y)\in A$, since A is closed under addition. By 2.2 (A, T_1) is a Peano succession whose generator is 1, and $1\in M$. Hence $M=A$. Since $f_x(y)\in A$ precludes $f_x(y)=0$, 6.2 is proved.

6.3. *If G is a subgroup and f an endomorphism, then $f(G)\subset G$.*

Proof. Like that of 5.3.

6.4. *If G and H are nontrivial subgroups (i.e., not $\{0\}$), then $G\cap H$ is a nontrivial subgroup.*

Proof. Let x and y be nonzero elements of G and H respectively. As in 5.4, using 6.3, $f_x(y)\in G\cap H$. By 6.2, $f_x(y)\neq 0$, so $G\cap H\neq\{0\}$.

Define a *dilation* of I as a homomorphism of a nontrivial subgroup of I into I. Endomorphisms are dilations, namely those with domain I. Addition of dilations has been defined as pointwise addition, and multiplication as composition. From 6.4 we are assured that if f and g are dilations, so are $f+g$ and $f\cdot g$, the argument for $f\cdot g$ being the same as that used in the corresponding situation in Section 5.

Equality in the neighborhood of 0 is a relation between dilations, written $f=(0)g$, defined thus: $f=(0)g$ if and only if $f(x)=g(x)$ for all x in some nontrivial subgroup. Again, the existence of some $x\neq 0$ such that $f(x)=g(x)$ is sufficient for $f=(0)g$.

6.5. *Equality in the neighborhood of* 0 *is an equivalence.*

Proof. Like 5.5.

6.6. *If $f_1=(0)f_2$ and $g_1=(0)g_2$, then $f_1+g_1=(0)f_2+g_2$ and $f_1\cdot g_1=(0)f_2\cdot g_2$.*

Proof. Like 5.6.

6.7. *For any dilation f there are endomorphisms g and h such that $f=(0)g\cdot h^{-1}$.*

Proof. Like 5.7.

Let the elements of R be the equivalence classes of dilations which are equal in the neighborhood of 0. For any dilation f, let $r(f)$ be the equivalence class to which it belongs. Define addition and multiplication in R by $r(f)+r(g)=r(f+g)$ and $r(f)\cdot r(g)=r(f\cdot g)$; this is possible by 6.6. In particular, let $0=r(f_0)$ and $1=r(f_1)$.

6.8. *R is a prime field.*

Proof. The Abelian group properties of R under addition come from the corresponding properties of I, via the corresponding properties for addition of dilations. Multiplication is associative because it comes from composition of dilations. The multiplicative identity is 1. By 6.7 any nonzero element of R contains a one to one dilation, say f, whose inverse is therefore a dilation; then $r(f^{-1})$ is $r(f)^{-1}$. The distributive laws for dilations go as follows: $(g+h)\cdot f(x)=g(f(x))+h(f(x))=(g\cdot f+h\cdot f)(x)$, and $(f\cdot(g+h))(x)=f(g(x)+h(x))=f(g(x))+f(h(x))=(f\cdot g+f\cdot h)(x)$ because f is a homomorphism. They carry over trivially to R. To show that R is prime, let F be a subfield, and let G be the set of x in I such that $r(f_x)\in F$. Then G is a subgroup of I containing 1, whence $G=I$. Thus $r(f)\in F$ for every endomorphism f, and by 6.7 $r(f)\in F$ for every dilation, that is $F=R$. By 4.1 R is a field.

6.9. *Define ϕ mapping I into R by $\phi(x)=r(f_x)$. Then ϕ is an isomorphism of I with the integer group of R. Hence R has characteristic zero.*

Proof. Like 5.9, except that proving ϕ is one to one is easier.

We have completed the proof of Theorem 3, and with it the construction of the rational numbers. Again, however, the following uniqueness theorem is worth having.

THEOREM 4. *Let F be a prime field of characteristic zero. Let I be the integer group of F, and let R be the field constructed from I in Theorem 3, ϕ the isomorphism of I with R_1. Then ϕ can be extended to an isomorphism of F with R.*

Proof. Like Theorem 2.

As a concluding remark we note that the process goes no further. A prime field has no endomorphisms except the identity, and no subfields except itself. What we have been exploiting in Peano successions and free cyclic groups is that they are "homogeneous" in the peculiar sense that each element is the beginning of a subsystem isomorphic with the whole parent system. Fields lack this flexibility.

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ON SETS OF ACQUAINTANCES AND STRANGERS AT ANY PARTY

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1. Introduction. In a recent issue of the MONTHLY, the following elementary problem [1] was posed:

*Prove that at a gathering of any six people, some three of them are either mutual acquaintances or complete strangers to each other.**

It is our purpose to prove a more general result when the number 6 is replaced by any positive integer N (see Theorem 1 below).

It is convenient to transform the problem into an equivalent problem concerning points and lines. The N persons involved are replaced by points A_k, $k=1, \cdots, N$, no three of which are collinear and if two persons are acquainted a line is drawn joining the corresponding pair of points. If the two persons are strangers then no line is drawn. Thus each collection of N people gives rise to a corresponding configuration of N points and L lines where $0 \leqq L \leqq N(N-1)/2$. If three people are mutually acquainted the corresponding figure is a triangle which we will call a *full triangle*. If three people are pairwise strangers the corresponding figure consists of three points with no lines joining any pair. We call such a figure an *empty triangle*. Any set of three points not the vertices of a full triangle, nor an empty triangle, will be called a *partial triangle*. Notice that a given point may simultaneously be a vertex of several triangles from each category. With these definitions we have

THEOREM 1. *Let E and F be the number of empty and full triangles respectively. Then in any configuration of N points*

$$E + F \geqq \begin{cases} \dfrac{u(u-1)(u-2)}{3}, & \text{if } N = 2u, \\ \dfrac{2u(u-1)(4u+1)}{3}, & \text{if } N = 4u+1, \\ \dfrac{2u(u+1)(4u-1)}{3}, & \text{if } N = 4u+3, \end{cases} \tag{1}$$

where u is a nonnegative integer, and this lower bound is sharp for each positive integer N.

We observe that for $N=6$, $u=3$ and then (1) gives $E+F \geqq 2$, which is a stronger result than the original problem suggested.

* The same problem in a different disguise appeared on the thirteenth William Lowell Putnam Examination in 1953: Six points are in general position in space (no three in a line, no four in a plane). The fifteen line segments joining them in pairs are drawn and then painted, some segments red, some blue. Prove that some triangle has all its sides the same color. In connection with other generalizations of this problem, see the article by Greenwood and Gleason [3].

2. The fundamental equations. Let p_j denote the number of points, each of which is a terminal point for exactly j of the lines. Then obviously

$$N = p_0 + p_1 + \cdots + p_L, \tag{2}$$

where $p_j \geqq 0$ for $j=0, 1, \cdots, L$. Since each of the L lines, joins two points a counting of the lines on each point gives $2L$; that is,

$$2L = p_1 + 2p_2 + \cdots + Lp_L. \tag{3}$$

Each line may be combined with each of the $N-2$ remaining points not on the line to form $N-2$ partial or full triangles. Hence there are $(N-2)L$ partial or full triangles, but in this counting some triangles have been counted more than once. Let us consider a particular point A at which j lines end. Any pair of these lines determines either a partial triangle with two sides, or a full triangle. The number of such partial triangles with two sides or full triangles with one vertex at A is $j(j-1)/2$, and for the entire configuration we have the sum

$$S = \sum_{j=2}^{L} \frac{j(j-1)}{2} p_j. \tag{4}$$

Now in the expression $(N-2)L$, a partial triangle with two sides has been counted twice and a full triangle has been counted three times. In the expression (4) each partial triangle with two sides is counted only once. However in (4) the full triangles are counted three times. Hence the expression

$$(N-2)L - \sum_{j=2}^{L} \frac{j(j-1)}{2} p_j \tag{5}$$

counts each partial triangle exactly once. Since the total number of triangles possible with N points is $N(N-1)(N-2)/6$ we have

LEMMA 1. *For any configuration of N points and L lines,*

$$E + F = \frac{N(N-1)(N-2)}{6} - (N-2)L + \sum_{j=2}^{L} \frac{j(j-1)}{2} p_j. \tag{6}$$

To minimize $E+F$, we consider the right side of (6). First for each fixed L we will minimize the sum S, and then we will determine a value for L which gives an absolute minimum for $E+F$.

LEMMA 2. *Let N and L be fixed with $0 \leqq L \leqq N(N-1)/2$. Then there is a set of nonnegative integers $(p_0, p_1, \cdots, p_L)$ which satisfies* (2) *and* (3) *and among such sets there is a unique one which makes S a minimum. For the minimizing set, at most two of the p_j are nonzero, and these two must be adjacent (subscripts differ by one). Further the minimizing set corresponds to a real configuration of lines and points.*

Proof. It is obvious that the system of equations (2) and (3) has at least one

solution in nonnegative integers because L lines can always be drawn. It follows immediately that some solution minimizes S. Let $(p_0, p_1, \cdots, p_L)$ be some solution in which two nonadjacent entries are positive. Indeed, let $p_j>0$ and $p_k>0$ with $j+d=k$, $d>1$. We construct a new solution $(p_0^*, p_1^*, \cdots, p_L^*)$ as follows. First suppose that $d>2$. Then set

$$
\begin{aligned}
p_i^* &= p_i, \quad \text{if} \quad i \neq j, j+1, k-1, k, \\
p_j^* &= p_j - 1, \qquad p_{k-1}^* = p_{k-1} + 1, \\
p_{j+1}^* &= p_{j+1} + 1, \qquad p_k^* = p_k - 1.
\end{aligned} \tag{7}
$$

It is obvious that this new set of nonnegative integers also satisfies (2) and (3). Further an easy computation shows that

$$S^* = S - (k - j - 1) = S - (d - 1) < S, \tag{8}$$

where S^* denotes the expression (4), evaluated for the new solution.

If $d=2$, so that $p_j>0$ and $p_{j+2}>0$, then we set

$$
\begin{aligned}
&p_i^* = p_i, \quad \text{if} \quad i \neq j, j+1, j+2, \\
&p_j^* = p_j - 1, \qquad p_{j+1}^* = p_{j+1} + 2, \qquad p_{j+2}^* = p_{j+2} -
\end{aligned} \tag{9}
$$

Again it is obvious that the set $(p_0^*, p_1^*, \cdots, p_L^*)$ satisfies (2) and (3) and that for this set

$$S^* = S - 1 < S. \tag{10}$$

Further, any solution of (2) and (3) corresponds to a realizable configuration, for it is easy to draw from each point the $2L$ half-lines as dictated by (3), assigning the proper number of half-lines to each point, and then join these half-lines pairwise to obtain L lines.

Thus starting with any solution, we can apply the two transformations described above stepwise until we arrive at a solution for which either $p_k>0$ and $p_{k+1}>0$ and the remaining p_j vanish, or a solution in which $p_k=N$ and the remaining p_j vanish. Finally the minimizing solution thus obtained does not depend on the initial solution chosen, because in any case where at most two of the p_j are nonzero the equation set (2) and (3) reduces to a pair of linear equations in at most two unknowns p_k and p_{k+1}. Solving these equations simultaneously gives

$$p_k = (k+1)N - 2L. \tag{11}$$

But then the index k is uniquely determined by (11) and the fact that $0<p_k \leqq N$. This completes the proof of Lemma 2. For example if $N=20$ and $L=44$ then by (11) $k=4$, $p_4=12$ and $p_5=8$ and this pair minimizes S for these values of L and N.

3. Determination of the absolute minimum. Let $\Delta(L)$ denote the change in the minimum of $E+F$ as L increases from L to $L+1$. Two cases must be considered. With L fixed we suppose that the solution which makes S a minimum has the form

$$\text{I. } p_k \geqq 2, \qquad p_{k+1} = N - p_k \geqq 0;$$
$$\text{II. } p_k = 1, \qquad p_{k+1} = N - 1.$$

When L is increased to $L+1$, the solutions which minimize S are respectively

$$\text{I. } p_k^* = p_k - 2, \qquad p_{k+1}^* = p_{k+1} + 2;$$
$$\text{II. } p_k^* = 0, \quad p_{k+1}^* = N - 1, \quad p_{k+2}^* = 1.$$

From the equation

$$\Delta(L) = -(N-2)(L+1) + (N-2)L + \sum_{j=2}^{L+1} (p_j^* - p_j)\frac{j(j-1)}{2}, \tag{12}$$

we find in Case I that

$$\Delta(L) = 2k + 2 - N, \tag{13}$$

and in Case II, that

$$\Delta(L) = 2k + 3 - N. \tag{14}$$

Now suppose that $N=2u$ is even. As L increases from 0 to $N(N-1)/2$, (13) and (14) show that $E+F$ first decreases, then is stationary, and then increases. Whence the minimum occurs when $k=u-1$ and whenever the conditions of Case I are satisfied. Thus the number of lines for a minimizing solution is not unique but is given by $2L=(u-1)p_{u-1}+up_u$, where p_{u-1} and p_u are the only nonzero elements among the p_j. However either p_{u-1} or p_u may be zero. Thus for L we have the limits $u^2-u \leqq L \leqq u^2$. To compute the minimum of $E+F$, we assume that $p_{u-1}=N$, whence $L=u^2-u$, $S=(u-1)(u-2)u$. Using these values in (6) gives the minimum announced in the theorem for N even.

Next suppose that $N=4u+1$. We observe that in Case I, $\Delta(L)$ is negative for $k=0, 1, \cdots, 2u-1$ and is positive thereafter. In Case II, $\Delta(L)=0$ for $k=2u-1$. Then $p_{2u-1}=1$, $p_{2u}=N-1$, and from (3), $2L=8u^2+2u-1$. But this is impossible because L must be an integer. Therefore the absolute minimum occurs in Case I, for the smallest index for which $\Delta(L)>0$. This gives $k=2u$, and $p_{2u}=N$, and hence $2L=2uN$, an even number. Then $S=2u(2u-1)(4u+1)/2$, and when these values are used in (6) we obtain $E+F=2u(u-1)(4u+1)/3$.

If $N=4u+3$, then Case II gives $\Delta(L)=0$ when $k=2u$. Thus one solution occurs when $p_{2u}=1$, $p_{2u+1}=N-1=4u+2$, $L=4u^2+5u+1$, and $S=8u^3+10u^2+u$. When these values are used in (6) we obtain $E+F=2u(4u-1)(u+1)/3$. A second minimizing solution occurs in this case, because $\Delta(L)=0$. For this second solution $p_{2u+1}=N-1$, $p_{2u+2}=1$, and $L=4u^2+5u+2$.

COROLLARY. *Let $E+F$ be a minimum for some configuration. Then*

$$u^2 - u \leqq L \leqq u^2, \qquad \text{if } N = 2u,$$

$$L = u(4u+1), \qquad \text{if } N = 4u+1,$$

$$(u+1)(4u+1) \leqq L \leqq (u+1)(4u+1)+1, \qquad \text{if } N = 4u+3.$$

Further if $N=4u+1$, each point lies on $2u$ lines. If $N=4u+3$, each point with one exception lies on $2u+1$ lines and the exceptional point lies either on $2u$ lines or on $2u+2$ lines.

4. Some open questions. It seems obvious that the methods used here should generalize to answer questions about the minimum number of full and empty quadrilaterals, and figures of a higher number of sides, but up to the present, I have not been able to carry through the computations successfully.

One could also ask if it is possible to have a configuration which minimizes $E+F$ in which either E or F is zero. In case $N=2u$ we can give an affirmative answer, but in the other cases, I have not been able to settle this question.

If C is a configuration, we construct C^*, a conjugate configuration, by drawing the line joining A_i and A_j in C^* if the line is missing in C, and by leaving the line out in C^* if they are joined by a line in C. Then empty triangles go into full triangles, partial triangles go into partial triangles, and full triangles go into empty ones in passing from C to C^*. It follows from this that if we wish to consider the problem of minimizing $E+F$ with $F=0$ or $E=0$ we need only examine one of the cases, say, $F=0$.

Suppose now that $N=2u$. We join points A_i and A_j if and only if i and j have different parity ($i-j$ is an odd integer). Then obviously there are u^2 lines in the resultant configuration. There are no full triangles because given any three points A_i, A_j, A_k, at least two must be of the same parity, and hence two of them are not joined by a line. The only empty triangles are those $A_iA_jA_k$ in which all three subscripts have the same parity. Therefore $E=u(u-1)(u-2)/3$. Thus in the configuration just described, $E+F$ is a minimum and $F=0$. The same type of configuration is not minimizing when N is odd.

Ramsey [4] has proved a very general existence theorem. Suppose that N points (or objects) are given and we form from these points all possible subsets consisting of k points, so that we have $\binom{N}{k}$ such subsets. These subsets are then distributed into r classes $C_1, \cdots, C_r$. If m is an integer, $m>k$, can we find m points such that all of the subsets of k of these m points fall in one of the classes C_i?

RAMSEY'S THEOREM. *There is a smallest integer $N_0=N_0(k, m, r)$ such that if $N \geqq N_0$, then for any distribution of the subsets of k points into r classes there is a set of m points for which all of its subsets of k points fall in one class.*

The function $N_0(k, m, r)$ is known as Ramsey's function, and the determination of this function, except in the simplest cases, is an open problem and a very

difficult one. The ideas which lie behind this theorem have been extended in a variety of directions, and one can find an account of these extensions and further references in an article by Erdös and Rado [2].*

The simplest case of Ramsey's theorem occurs when pairs of points are distributed into two classes ($k=2$, $m=2$) and we are to find $m=3$ points such that all pairs of these 3 points are in one of the classes. As the problem proposed by Bostwick shows, this will always occur when $N \geqq 6$, and it is easy to construct a distribution of the pairs when $N=5$, for which the property fails. This shows that $N_0(2, 3, 2)=6$.

If $k=2$, it is convenient to consider the pairs of points as connected by lines, and these lines painted with a suitable color corresponding to the particular class C_i ($i=1, \cdots, r$) in which it falls. When this is done the result is a chromatic graph. Greenwood and Gleason [3] have studied these chromatic graphs and have proved that $N_0(2, 4, 2)=18$ and $N_0(2, 3, 3)=17$. As far as the author is aware these three cases are the only nontrivial cases for which the Ramsey function has been evaluated, although various estimates have been obtained [2].

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* I am indebted to John Isbell for calling my attention to this reference. I am also indebted to the referee for calling my attention to the article by Greenwood and Gleason [3].

SPHÈRES ASSOCIÉES À UN TÉTRAÈDRE†

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Notations. Soient un tétraèdre $T \equiv ABCD$ inscrit à une sphère (O), de centre O; a, a', b, b', c, c' et A, B, C, D les mesures des arêtes BC, DA, CA, DB, AB, DC dont les milieux sont $A_1, A_1', B_1, B_1', C_1, C_1'$, et celles des aires des faces BCD, CDA, DAB, ABC; (DA) la sphère tangente en D à la face BCD et passant par les sommets D et A; (D) le terne de sphères (DA), (DB), (DC); (A), (B), (C) les ternes de sphères analogues à (D) associées aux sommets A, B, C; $(BCD\text{-}DA)$

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la sphère tangente en D à l'arête DA et circonscrite à la face BCD; ($\mathfrak{D}$) le terne des sphères [(DBC-DA), (DCA-DB), (DAB-DC)]; ($\mathfrak{A}$), ($\mathfrak{B}$), ($\mathfrak{C}$) les ternes de sphères analogues à ($\mathfrak{D}$) et associées aux ternes d'arêtes (AB, AC, AD), (BA, BC, BD), (CA, CB, CD).

1. THÉORÈME. *Les axes radicaux des ternes de sphères* (A), (B), (C), (D) *se confondent avec ceux des ternes* ($\mathfrak{A}$), ($\mathfrak{B}$), ($\mathfrak{C}$), ($\mathfrak{D}$) *et forment un quadruple hyperholoïdique.*

Démonstration. Une inversion i, de pôle D, transforme les sommets A, B, C en des points A', B', C' et les sphères (DA), (DB), (DC) en des plans menés par A', B', C' parallélement aux faces opposées de T. Ces plans coupent la droite joignant D au barycentre g du triangle $A'B'C'$ en un point g' tel que $gg'=2\,Dg$, et la droite Dg coincide avec l'axe radical du terne (D). Les sphères (BCD-DA), (CDA-DB), (DAB-DC) se transforment en les plans parallèles aux arêtes du trièdre D-ABC menés par $B'C'$, $C'A'$, $A'B'$ qui concourent sur Dg en un point g'' tel que $gg''=Dg/2$. Les axes radicaux des ternes (D) et ($\mathfrak{D}$) sont donc confondus ainsi que, par analogie, ceux de (A) et ($\mathfrak{A}$), (B) et ($\mathfrak{B}$), (C) et ($\mathfrak{C}$). D'autre part, les plans (DA, Dg), (DB, Dg), (DC, Dg) passent par une médiane des triangles $DB'C'$, $DC'A'$, $DA'B'$, autrement dit, par une symédiane des triangles DBC, DCA, DAB, puisque (BC, $B'C'$), (CA, $C'A'$), (AB, $A'B'$) sont des couples antiparallèles pour ces triangles. La droite Dg rencontre donc les céviennes, par rapport à T, des points de *Lemoine* des faces DBC, DCA, DAB et par suite les céviennes des points de *Lemoine* des faces de T. Ces quatre céviennes étant rencontrées par la droite Dg et ses trois analogues, les huit droites appartiennent à un même hyperboloïde.

Autrement, l'axe radical du terne (D) rencontre le plan ABC au point D_2 de coordonnées barycentriques ($1/a'^2$, $1/b'^2$, $1/c'^2$), pour le triangle ABC [1]. Or, les distances des points de l'axe radical du terne ($\mathfrak{D}$) aux plans DBC, DCA, DAB étant proportionnelles aux quantités ($1/A\cdot a'^2$, $1/B\cdot b'^2$, $1/C\cdot c'^2$) [2], les axes radicaux de (D) et ($\mathfrak{D}$) coïncident. D'autre part, d'après les coordonnées de D_2 et des points analogues A_2, B_2, C_2, sur BCD, CDA, DAB, les droites AA_2, BB_2, CC_2, DD_2 sont hyperboloïdiques [3].

Remarques. L'axe radical $Dg\equiv DD_2$ des sphères (D) est le lieu des barycentres des sections antiparallèles de T relatives au trièdre D-ABC, sections parallèles au plan tangent en D à (O). Il rencontre respectivement les sphères (D), les sphères ($\mathfrak{D}$), aux transformés des points g', g'', g par i et situés à des distances de D proportionnelles à 1, 2, 3. Les centres des sphères (D), ($\mathfrak{D}$) coïncident avec six sommets d'un parallélpipède dont O et D sont les autres sommets opposés. Le trièdre de sommet D de ce parallélipipède est le supplémentaire du trièdre D-ABC. Le centre d'une sphère de (D) coïncide avec le symétrique de O par rapport au milieu de la distance des centres de deux sphères de ($\mathfrak{D}$) et vice versa.

COROLLAIRE. *Si le tétraèdre est isodynamique, les axes radicaux* AA_2, BB_2, CC_2, DD_2 *concourent au second point de Lemoine de* T.

Démonstration. A_2, B_2, C_2, D_2 coïncident avec les points de *Lemoine* des faces BCD, CDA, DAB, ABC [4].

2. THÉORÈME. *Les axes radicaux des ternes de sphères* $[(BA), (CA), (DA)]$, $[(CB), (DB), (AB)]$, $[(DC), (AC), (BC)]$, $[(AD), (BD), (CD)]$ *coïncident avec les droites qui joignent les sommets de* T *aux centres des cercles circonscrits aux faces opposées et forment un quadruple hyperboloïdique si le tétraèdre* T *est équifacial, orthocentrique ou bisymétrique.*

Démonstration. Les sphères (AD), (BC), (CD) étant tangentes au plan ABC en A, B, C le centre O_d du cercle ABC a même puissance pour ces cercles, et la droite DO_d est l'axe radical des trois sphères. Même conclusion pour les droites AO_a, BO_b, CO_c et les ternes de sphères correspondantes. D'autre part, les droites qui joignent les sommets d'un tétraèdre non dégénéré aux centres des cercles circonscrits aux faces opposées sont hyperboloïdiques si, et seulement si le tétraèdre fondamental T est équifacial, orthocentrique ou bisymétrique [5].

3. THÉORÈME. *Le centre radical de quatre sphères passant par trois sommets et tangentes à une même arête ou à celle qui lui est opposée est situé sur la bimédiane relative à ces deux arêtes opposées de* T.

Démonstration. Il est clair, par exemple, que le plan radical π des sphères $(BCD\text{-}DA)$, $(ABC\text{-}AD)$ passe par l'arête BC et le milieu A_1' de la tangente DA à ces sphères, et que celui des sphères $(DAB\text{-}BC)$, $(CDA\text{-}CB)$ passe par DA et A_1. Le centre radical E_1 de ces quatre sphères est donc sur la bimédiane A_1A_1' qu'il divise dans le rapport $-a'^2/a^2$, car, d'après le théorème classique relatif à la différence des puissances d'un point par rapport à deux sphères, on constate que π divise l'arête CA dans ce rapport $-a'^2/a^2$. Or, les plans passant par l'arête DB coupent CA et A_1A_1' suivant des divisions semblables puisque le points à l'infini se correspondent.

Autrement, A, B, C, D ayant pour puissances a'^2, a^2, a^2, a'^2, par rapport aux sphères $(BCD\text{-}DA)$, $(CDA\text{-}CB)$, $(DAB\text{-}BC)$, $(ABC\text{-}AD)$, le centre radical E_1 de ces sphères coincide avec le point de coordonnées barycentriques $(1/a'^2, 1/a^2, 1/a^2, 1/a'^2)$ [2]. De même, les centres radicaux E_2, E_3 des sphères $(BCD\text{-}CA)$, $(CDA\text{-}DB)$, $(DAB\text{-}AC)$, $(ABC\text{-}BD)$ et $(BCD\text{-}BA)$, $(CDA\text{-}AB)$, $(DAB\text{-}DC)$, $(ABC\text{-}CD)$, de coordonnées $(1/b^2, 1/b'^2, 1/b^2, 1/b'^2)$ et $(1/c^2, 1/c^2, 1/c'^2, 1/c'^2)$, sont sur les bimédianes B_1B_1' et C_1C_1' qu'ils divisent dans les rapports $-b'^2/b^2$ et $-c'^2/c^2$.

4. THÉORÈME. *Les centres radicaux des trois groupes de quatre sphères tangentes à une face de* T *et passant par les deux sommets d'une même arête ou de son opposée, sont trois points alignés situés sur les côtés du triangle déterminé par les centres radicaux des trois groupes de quatre sphères passant par trois sommets de* T *et tangentes à une même arête ou à son opposée* (J. Hecquet).

Démonstration analytique. Pour faciliter l'écriture, modifions les notations en considérant un tétraèdre $T \equiv A_iA_jA_kA_l$, d'arêtes $A_iA_j = a_{ij}, \cdots$. Soient (ij) la

sphère passant par A_i et A_j et tangente en A_i à la face $A_iA_kA_l$; (ikl) la sphère passant par A_i, A_k, A_l et tangente en A_i à l'arête A_iA_j. En se rapportant a un système de coordonnées barycentriques pour T, et posant

$$S = \sum \overset{2}{a}_{ij}x_ix_j \text{ (sphère } A_iA_jA_kA_l), \qquad I = \sum x_i \text{ (plan de l'infini)},$$

la sphère (ikl) a pour équation

$$-S + I \cdot \overset{2}{a}_{ij}x_j = 0,$$

et le centre radical e_1 des sphères (ikl), (jkl), (kij), (lij) de coordonnées $x_i = x_j = a_{kl}^2$, $x_k = x_l = a_{ij}^2$, coïncide avec le centre de la quadrique

$$Q_{ij} \equiv \overset{2}{a}_{kl}u_iu_j + \overset{2}{a}_{ij}u_ku_l = 0.$$

D'autre part, la sphère (ij) a pour équation

$$-S + I(\overset{2}{a}_{ik}x_k + \overset{2}{a}_{il}x_l) = 0,$$

et le centre radical f_1 des sphères (ij), (ji), (kl), (lk), qui a pour coordonnées

$$x_i = \overset{2}{a}_{jl} - \overset{2}{a}_{jk}, \qquad x_j = \overset{2}{a}_{ik} - \overset{2}{a}_{il}, \qquad x_k = \overset{2}{a}_{jl} - \overset{2}{a}_{il}, \qquad x_l = \overset{2}{a}_{ik} - \overset{2}{a}_{jk},$$

coïncide avec le centre de la quadrique $q_{ij} \equiv Q_{ik} - Q_{il} = 0$, situé sur la droite des centres des quadriques Q_{ij}, Q_{il}. Les trois quadriques q_{ij}, q_{ik}, q_{il} qui correspondent aux sphères (ij), (ik), (il) satisfont à la condition

$$q_{ij} + q_{ik} + q_{il} = 0,$$

et appartiennent à un faisceau tangentiel; leurs centres f_1, f_2, f_3 sont donc colinéaires et situés sur les droite des centres e_1, e_2, e_3 de Q_{ij}, Q_{ik}, Q_{il}.

COROLLAIRE. *Le centre de la quadrique tangente aux quatres faces de T en leurs points de Lemoine est situé dans le plan $p \equiv (e_1, e_2, e_3)$.*

Démonstration. Le centre de cette quadrique d'équation

$$Q_{ij} + Q_{ik} + Q_{il} \equiv \sum \overset{2}{a}_{ij}u_ku_l = 0$$

est dans le plan des centres de Q_{ij}, Q_{ik}, Q_{il}, lieu des centres des quadriques du réseau tangentiel déterminé par ces trois quadriques.

COROLLAIRE. *Si le tétraèdre T est équifacial, les centres radicaux e_1, e_2, e_3, f_1, f_2, f_3 sont confondus avec le barycentre de T.*

COROLLAIRE. *Le centre de la quadrique tangente aux quatre faces de T en leurs points de recontre avec les axes radicaux des ternes de sphères définis au théorème* (Sec. 1), *est situé dans le plan p.*

Démonstration. L'axe radical des sphères (ikl), (ijl), (ijk) a pour équations $a_{ij}^2 x_j = a_{ik}^2 x_k = a_{il}^2 x_l$ et sa trace sur $x_i = 0$ pour coordonnées

$$x_i = 0, \qquad x_j = 1/a_{ij}^2, \qquad x_k = 1/a_{ik}^2, \qquad x_l = 1/a_{il}^2.$$

Or, la quadrique $Q_{ij} = 0$ peut aussi s'écrire

$$Q'_{ij} \equiv u_i u_j / a_{ij}^2 + u_k u_l / a_{kl}^2 = 0,$$

et le centre de la quadrique

$$Q'_{ij} + Q'_{ik} + Q'_{il} \equiv \sum u_i u_j / a_{ij}^2 = 0,$$

tangente aux faces de T aux quatre points en cause, est situé dans le plan p.

5. Points de Brocard. Si l'on marque des points arbitraires a, b, c sur les côtés BC, CA, AB d'un triangle t, les cercles Abc, Bca, Cab concourent en un point D. Lorsque a, b, c coïncident respectivement avec les sommets de t, ce qui donne

$$c \equiv A,\ b \equiv C,\ a \equiv B \quad \text{ou} \quad c \equiv B,\ b \equiv A,\ a \equiv C,$$

les cercles Abc, Bca, Cab sont tangents en A, B, C à AB, BC, CA ou CA, AB, BC, et ces ternes de cercles concourent aux points de *Brocard* de t. Par analogie avec cette figure plane, après avoir constaté qu'une inversion dont le pôle est extérieur au plan ABC transforme celui-ci en une sphère et les droites AB, BC, CA en des cercles concourants, si l'on choisit arbitrairement un point α, α', β, β', γ, γ' sur chaque arête BC, DA, CA, DB, AB, DC d'un tétraèdre T, les sphères $A\beta\gamma\alpha'$, $B\gamma\alpha\beta'$, $C\alpha\beta\gamma'$, $D\alpha'\beta'\gamma'$ ont un point commun Q. (sphères de *Miquel*, [6]). Si quatre sphères prises dans l'un ou l'autre ou dans les deux groupes $[(A), (B), (C), (D)]$, $[(\mathfrak{A}), (\mathfrak{B}), (\mathfrak{C}), (\mathfrak{D})]$, sont des sphères de *Miquel*, un ou deux points $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$ coïncident nécessairement avec des sommets de T. On constate aisément que les sphères (DA), (AB), (BC), (CD) ne forment pas un quaterne de *Miquel*, pas plus que les sphères $(ABC\text{-}AD)$, $(BCD\text{-}BA)$, $(CDA\text{-}CB)$, $(DAB\text{-}DC)$. Mais dans l'hypothèse selon laquelle

$$\alpha \equiv C,\ \ \alpha' \equiv A,\ \ \beta \equiv A,\ \ \beta' \equiv D,\ \ \gamma \equiv B,\ \ \gamma' \equiv D,$$

les sphères (DA), (AB), $(BCD\text{-}BA)$, $(CDA\text{-}CB)$ forment un quaterne de *Miquel*, car

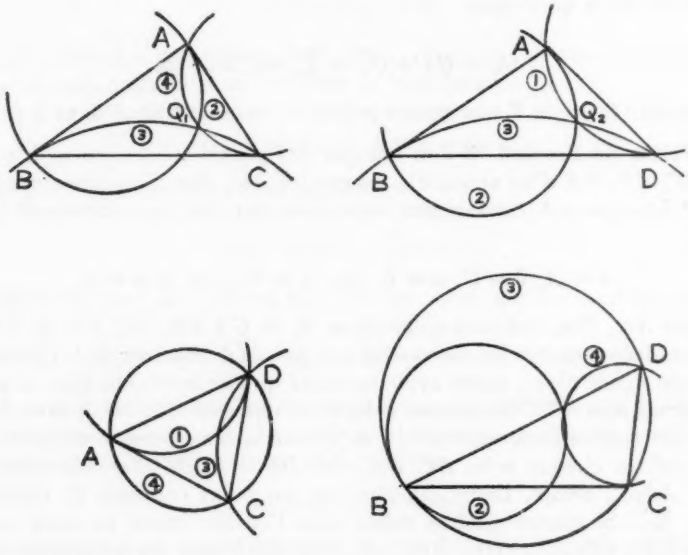
$$\left.\begin{matrix}(DA)\\(AB)\\(BCD\text{-}BA)\\(CDA\text{-}CB)\end{matrix}\right\} \text{coupe les arêtes} \left\{\begin{matrix}DA,\ DB,\ DC\\AD,\ AB,\ AC\\BA,\ BD,\ BC\\CB,\ CA,\ CD\end{matrix}\right\} \text{en} \left\{\begin{matrix}A,\ D,\ D,\\A,\ B,\ A,\\B,\ D,\ C,\\C,\ A,\ D.\end{matrix}\right.$$

De même

$$\left.\begin{matrix}(AD)\\(BA)\\(ABC\text{-}CD)\\(BCD\text{-}DA)\end{matrix}\right\}\ \text{coupe les arêtes}\ \left\{\begin{matrix}AD,\ AB,\ AC\\BA,\ BC,\ BD\\CA,\ CB,\ CD\\DA,\ DB,\ DC\end{matrix}\right\}\ \text{en}\ \left\{\begin{matrix}D,\ A,\ A,\\A,\ B,\ B,\\A,\ B,\ C,\\D,\ B,\ C,\end{matrix}\right.$$

et ces sphères forment un quaterne de *Miquel* avec

$$\alpha \equiv B,\quad \alpha' \equiv D,\quad \beta \equiv A,\quad \beta' \equiv B,\quad \gamma \equiv A,\quad \gamma' \equiv C.$$



Dans chacune de ces hypothèses, les quatres sphères considérées concourent en un point Q comparable à l'un des points de *Brocard* d'un triangle. En associant ainsi les sphères des deux groupes, on obtient 24 quaternes de sphères de *Miquel* auxquels correspondent 24 points Q.

Construction d'un point Q. Si l'on pose $(DA) \equiv ①$, $(AB) \equiv ②$, $(BCD\text{-}BA) \equiv ③$, $(CDA\text{-}CB) \equiv ④$, l'examen des figures permet de constater que les sections des sphères [②, ③, ④] et [②, ③, ①] par les plans ABC et DAB déterminent, grâce à ③, des cercles qui se coupent en l'un des points Q_1 et Q_2 de *Brocard* des faces ABC et DAB. Le point Q commun aux quatre sphères en cause coïncide donc avec l'image commune de Q_1 et Q_2 par rapport aux plans des centres des sphères ②, ③, ④ et ②, ③, ①. Si l'on choisit deux faces de T, choix possible de six manières, et un point de *Brocard* dans l'une quelconque de ces deux faces, trois des quatre sphères sont déterminées et la quatrième peut être choisie de deux manières, ce qui porte bien à 24 le nombre des points Q.

Références

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MATHEMATICAL NOTES

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INTEGRAL TRANSFORMATIONS AND CLOSED-FORM EXPRESSIONS FOR SUMS OF INFINITE SERIES

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Wheelon [3] has presented an interesting application of the Laplace transformation in obtaining closed form expressions for the sums of certain infinite series. Since the volumes by Erdélyi et al. [2] include extensive tables of transformation pairs other than those of the Laplace type some of these others might be useful.

Consider, in particular, the Mellin transformation

$$g(s) = \int_0^\infty x^{s-1} f(x)\,dx.$$

If we set $s = n+1$, multiply by coefficients a_n, and sum on n; then, provided that it is permissible to interchange the operations of summation and integration (for example, see Bromwich [1]),

$$\sum_{n=0}^{\infty} a_n g(n+1) = \int_0^\infty \left\{ \sum_{n=0}^{\infty} a_n x^n \right\} f(x)\,dx.$$

Thus, if the terms of the desired series can be expressed as a product of the coefficients of a known power series and of Mellin transforms, the series can be

expressed as a definite integral and tables will perhaps lead to a closed form expression.

With certain specific choices for the a_n we can reduce the integral on the right to standard transformation integrals (sometimes a simple change of variable is also required) as, for example:

(1) $a_n=(-b)^n/n!$, Laplace transformation,

$$\sum_{n=0}^{\infty}(-b)^n g(n+1)/n! = \int_0^{\infty} e^{-bx}f(x)dx;$$

(2) $a_n=(-b^2)^n/(2n)!$, Fourier cosine transformation,

$$\sum_{n=0}^{\infty}(-b^2)^n g(n+1)/(2n)! = \int_0^{\infty} \cos bu\{2uf(u^2)\}du;$$

(3) $a_n=(-b^2)^n/2^{2n}(n!)^2$, Hankel transformation of order zero,

$$\sum_{n=0}^{\infty}(-b^2)^n g(n+1)/2^{2n}(n!)^2 = b^{-1/2}\int_0^{\infty} J_0(bu)(bu)^{1/2}\{2u^{1/2}f(u^2)\}du;$$

(4) $a_n=(-1)^n/b^n$, Stieltjes transformation,

$$\sum_{n=0}^{\infty}(-1)^n g(n+1)/b^n = b\int_0^{\infty}(b+x)^{-1}f(x)dx;$$

In each case the g-function is the Mellin transform of the f-function.

A specific example will illustrate the technique. Consider the sum

$$\sum_{n=0}^{\infty}(-2c)^n\Gamma\{(n-\nu+1)/2\}\Gamma\{(n+\nu+1)/2\}/\Gamma(n+1) = \sum_{n=0}^{\infty}(-c)^n g(n+1)/n!$$

with the terms factored as indicated and where the Mellin transform is chosen as $g(s)=2^{s-1}\Gamma\{(s-\nu)/2\}\Gamma\{(s+\nu)/2\}$, so that from [2], 6.8(26), $f(x)=2K_\nu(x)$, and the integral becomes

$$2\int_0^{\infty} e^{-cx}K_\nu(x)dx.$$

This is a Laplace integral, hence from [2], 4.16(24) the value of the sum for $|\nu|<1$ is given as

$$\pi \csc(\nu\pi)(c^2-1)^{-1/2}\{[c+(c^2-1)^{1/2}]^{\nu}-[c+(c^2-1)^{1/2}]^{-\nu}\}.$$

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A NOTE ON FUNCTIONS CONTINUOUS ALMOST EVERYWHERE

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Let $f: R \to R$ be a single valued function of a real variable, R being the set of real numbers. f is termed a *CAE* if and only if f is continuous almost everywhere. We give now a necessary and sufficient condition for f to be a *CAE*.

THEOREM 1. *$f: R \to R$ is a CAE if and only if for O^* open, then $f^{-1}(O^*)$ can be written as a union $O \cup A$, where O is open and $m(A) = 0$. $m(A)$ denotes the Lebesgue measure of A.*

Proof. Necessity. Let $f: R \to R$ be a *CAE*. Let O^* be open in R and $X = f^{-1}(O^*)$. Set $X = X_c \cup X_d$ where X_c is the set of points of continuity of f in X and X_d is the set of points of discontinuity of f in X. Let $x \in X_c$. Then there exists O_x such that $x \in O_x$ and $f(O_x) \subset O^*$. Then $O_x \subset f^{-1}(O^*)$. Hence $X = X_c \cup X_d \subset \bigcup_{x \in X_c} O_x \cup X_d \subset X$. Thus $X = \bigcup_{x \in X_c} O_x \cup X_d$, where $m(X_d) = 0$ and $\bigcup_{x \in X_c} O_x$ is open.

Sufficiency. Let E be the set of points of discontinuity of f. Let $p \in E$. Then there exists an O_p^* open such that $f(p) \in O_p^*$ and $p \in O$ implies that $f(O) \not\subset O_p^*$. There exists a neighborhood $N(s; r)$ with rational center s and positive rational radius r such that $f(p) \in N(s; r) \subset O_p^*$. Then $p \in O$ implies $f(O) \not\subset N(s; r)$. But $f^{-1}(N(s; r)) = O_{sr} \cup A_{sr}$, where O_{sr} is open and A_{sr} has measure zero. Now $p \in f^{-1}(N(s; r))$ and $p \notin O_{sr}$. Thus $p \in A_{sr}$. Hence $E \subset \bigcup_{r,s} A_{sr}$ and thus $m(E) = 0$.

COROLLARY. *$f: R \to R$ is CAE if and only if F^* closed implies $f^{-1}(F^*)$ can be written as $F - A$, where F is closed and $m(A) = 0$.*

Proof. Sufficiency. Let O^* be open. Then $f^{-1}(-O^*) = F - A$, where F is closed and $m(A) = 0$. Taking complements we have $-f^{-1}(-O^*) = -F \cup A$ or $f^{-1}(O^*) = O \cup A$, where $O = -F$ and thus is open. This proves that f is a *CAE*.

Necessity. Let F^* be closed and f a *CAE*. Then $f^{-1}(F^*) = -f^{-1}(-F^*) = -(O \cup A)$, where O is open and $m(A) = 0$. Thus $f^{-1}(F^*) = (-O) \cap (-A) = (-O) - A$. Since $-O$ is closed, the necessity is proved.

A *CAE* of a *CAE* is not in general a *CAE*. For consider the following example: Let

$$g(x) \equiv \begin{cases} 0 \text{ if } x \text{ is irrational,} \\ 1/p \text{ if } x = q/p \text{ where } q, p \text{ are relatively prime integers and } p \text{ positive.} \end{cases}$$

It is well known that g is a *CAE*. Now let

$$f(x) \equiv \begin{cases} 1 \text{ if } x = 1/n \text{ for } n = 1, 2, \cdots, \\ 0 \text{ otherwise.} \end{cases}$$

It is also obvious that f is a *CAE*. But the composite fg is 1 if x is rational and 0 if x is irrational. Thus fg is not a *CAE*.

Suppose f and g are both *CAE* and let $h = fg$. Then $h^{-1}(O^*) = g^{-1}(f^{-1}(O^*)) = g^{-1}(O' \cup A') = g^{-1}(O') \cup g^{-1}(A') = O \cup A \cup g^{-1}(A')$, where O, O', and O^* are

open and A and A' are sets of measure zero. Then h is a CAE if and only if the set $O \cup A \cup g^{-1}(A')$ can be written as the union of an open set and a set of measure zero. The following theorems are then easy consequences of the above remarks.

THEOREM 2. *Let f and g be CAE, and $m(g(A))>0$ whenever $m(A)>0$. Then $h=fg$ is CAE.*

THEOREM 3. *Let f be continuous and g be CAE. Then $h=fg$ is CAE.*

We conclude with the remark that the set of CAE's is a ring which might have some interesting properties.

AN APPLICATION OF MATRICES TO LINEAR RECURSION RELATIONS*

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Matrix methods are helpful in solving certain problems stemming from linear recursion relations, such as that of finding an explicit expression for the nth term of the Fibonacci sequence, or of analyzing the vibration of a weighted string ([1] pp. 152–154). The procedure will be illustrated by means of a simple example.

Suppose that a sequence of complex numbers, $\{a_n\}$, is defined recursively as follows:

$$a_0 = a,\ a_1 = b,\ a_{n+2} = p \cdot a_{n+1} + q \cdot a_n,\ n \geqq 0\ (a, b, p, q \text{ complex}).$$

What is an explicit expression for a_n in terms of a, b, p, q, n?

We note that

$$\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = \begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}^2 \begin{bmatrix} a_{n-1} \\ a_{n-2} \end{bmatrix} = \cdots = \begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} b \\ a \end{bmatrix}.$$

Let

$$R = \begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}.$$

The characteristic equation of R is $k^2 - pk - q = 0$.

Case 1. $p^2+4q \neq 0$. In this case, R has two distinct characteristic roots, k_1, k_2, say. Then there exists a matrix A such that

$$A^{-1}RA = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}.$$

It is clear that $(A^{-1}RA)^n = A^{-1}R^nA$. Therefore

* Presented at a meeting of the Northeastern Section of the Mathematical Association of America at Durham, N. H., on Nov. 25, 1955.

$$R^n = A\begin{bmatrix} k_1^n & 0 \\ 0 & k_2^n \end{bmatrix} A^{-1}.$$

Hence

$$\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = A\begin{bmatrix} k_1^n & 0 \\ 0 & k_2^n \end{bmatrix} A^{-1}\begin{bmatrix} b \\ a \end{bmatrix}.$$

Since the elements of A are independent of n, we can conclude that $a_n = c_1 \cdot k_1^n + c_2 \cdot k_2^n$, where c_1, c_2 are complex numbers independent of n. We can determine the values of c_1, c_2 by setting $n=0$, 1: $c_1+c_2=a$, $c_1k_1+c_2k_2=b$. These two equations, together with the values of k_1, k_2 obtained from the characteristic equation, give the desired result.

Case 2. $p^2+4q=0$. In this case, R has a repeated characteristic root, k_0, say. There exists a matrix A such that

$$A^{-1}RA = \begin{bmatrix} k_0 & 0 \\ 1 & k_0 \end{bmatrix}.$$

Hence

$$\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = \begin{bmatrix} k_0^n & 0 \\ nk_0^{n-1} & k_0^n \end{bmatrix} A^{-1}\begin{bmatrix} b \\ a \end{bmatrix}.$$

By arguing as in Case 1, one obtains $a_n = c_1k_0^n + c_2 \cdot n \cdot k_0^{n-1}$, where $c_1 = a$ and $c_2 = b - a \cdot k_0$.

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THE DIMENSION OF THE MAGIC SQUARE VECTOR SPACE*

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It was stated in [1] that the set of all magic squares of order n forms a linear subspace of the algebra of semimagic squares. We became interested in determining the dimension of this space. Our solution, given below, involves the characteristic of the field in a rather interesting way.

Let F be a field. For matrices A, $B \in F_n$, define $(A|B)$ to be the trace of AB^T. Regarding F_n as an F-vector space, we wish to determine the dimension of the subspace M_n of those $M \in F_n$ for which

$$(1) \qquad (C_i|M) = (R_i|M) = (I|M) = (J|M) \qquad (i = 1, \cdots, n),$$

* This note is extracted from a Master's thesis written at the State University of Iowa. The present proof is due to suggestions of Professors S. K. Berberian and M. F. Smiley. The original proof, as given in the thesis, is somewhat less elegant but has the advantage of providing an explicit basis for M_n.

where $C_i(R_i)$ is the matrix of F_n whose ith column (row) has all entries 1 and whose other entries are zero, I is the identity matrix of F_n, and J is the matrix of F_n such that $a_{ij}=1$ for $i+j=n+1$ and $a_{ij}=0$ otherwise $(i, j=1, \cdots, n)$. Elements of M_n are called *F-magic squares*. Since the cases $n=1, 2$ are trivial, we restrict our discussion to $n \geqq 3$.

LEMMA 1. *If $n \geqq 3$, then $C_1, \cdots, C_n, R_1, \cdots, R_{n-1}, I, J$ are F-linearly independent unless $n=3$ and $3=0$ in F or $n=4$ and $2=0$ in F.*

Proof. Assume that

$$\alpha_1 C_1 + \cdots + \alpha_n C_n + \beta_1 R_1 + \cdots + \beta_{n-1} R_{n-1} + \gamma I + \delta J = 0. \tag{2}$$

The $(n, 2), \cdots, (n, n-1)$ components in (2) yield $\alpha_2 = \cdots = \alpha_{n-1} = 0$ and the $(1, 2)$ component then gives $\beta_1 = 0$. The $(1, 1)$, $(n, 1)$, and (n, n) components in (2) then give $\alpha_1+\gamma=\alpha_1+\delta=\alpha_n+\gamma=0$, and it follows that $\alpha_1=\alpha_n=-\gamma=-\delta$. Since $\alpha_1=0$ then implies that $\beta_2=\cdots=\beta_{n-1}=0$, we may as well suppose that $\alpha_1=\alpha_n=1$, $\gamma=\delta=-1$ and reach a contradiction. If $n>4$, then the $(2, 3)$ and $(2, 2)$ components in (2) yield $\beta_2=\beta_2-1=0$, a contradiction. If $n=4$, the $(2, 1)$ and $(2, 2)$ components in (2) give $\beta_1=-1$, $\beta_2-1=-2=0$, a contradiction. If $n=3$, then the second row of (2) reduces to $(\beta_2+1, \beta_2-2, \beta_2+1)=0$, and we see that $\beta_2=-1$, $\beta_2-2=-3=0$, a contradiction.

We state the following two lemmas without proof. They may be proved by the method used to prove Lemma 1.

LEMMA 2. *Let $n=3$ and $3=0$ in F. Then $C_1, C_2, C_3, R_1, R_2, I$ are F-linearly independent and $J=C_1+C_3-I-R_2$.*

LEMMA 3. *Let $n=4$ and $2=0$ in F. Then $C_1, C_2, C_3, C_4, R_1, R_2, R_3, I$ are F-linearly independent and $J=C_1+C_4+R_2+R_3+I$.*

We are now in a position to prove the following

THEOREM. *If $n \geqq 3$, the dimension of M_n is n^2-2n unless $n=4$ and $2=0$ in F when this dimension is n^2-2n+1.*

Proof. Case I: The hypotheses of Lemmas 2 and 3 are false. Define $B_i=C_i$ $(i=1, \cdots, n)$, $B_{j+n}=R_j$ $(j=1, \cdots, n-1)$, $B_{2n}=I$, $B_{2n+1}=J$, $D_k=B_k-B_{k+1}$ $(k=1, \cdots, 2n)$. By Lemma 1, it is clear that $D_1, \cdots, D_{2n}$ are F-linearly independent. It is also clear that the equations (1) are equivalent to

$$(D_k \mid M) = 0 \qquad (k = 1, \cdots, 2n). \tag{3}$$

Regard each D_k, and M as well, as a vector in $V_{n^2}(F)$. Let Δ be the $2n$ by n^2 matrix obtained by arranging each D_k as a row vector by lining up its rows (in order) into a row. Then Δ has rank $2n$ and its null space, $\mathfrak{M}_n$, has dimension n^2-2n.

Case II: $n=3$ and $3=0$ in F. Since $J=C_1+C_3-I-R_2$ the equations (1) imply that $(J \mid M)=0$. By Lemma 2, the equations (1) are equivalent to

$$(C_1 \mid M) = (C_2 \mid M) = (C_3 \mid M) = (R_1 \mid M) = (R_2 \mid M) = (I \mid M) = 0.$$

An argument like that of Case I then gives 9-6 as the dimension of $\mathfrak{M}_3$.

Case III: $n=4$ and $2=0$ in F. Here $J=C_1+C_4+R_2+R_3+I$ and $(J \mid M)$ is arbitrary in F. With $B_i=C_i$ $(i=1, \cdots, 4)$, $B_{j+4}=R_j$ $(j=1, 2, 3)$, $B_8=I$, $D_k=B_k-B_{k+1}$ $(k=1, \cdots, 7)$, we see by Lemma 3 that the equations (1) are equivalent to

$$(D_k \mid M) = 0 \qquad (k = 1, \cdots, 7). \tag{4}$$

As in Case I, M_n has dimension $n^2-7=n^2-2n+1$.

Reference

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A NOTE ON ALGEBRAS

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An *n-dimensional associative algebra* A over a field F is an (associative) ring which is simultaneously an n-dimensional vector space over F, and in which the ring multiplication and scalar multiplication are connected by

$$x(cy) = c(xy) = (cx)y \qquad (c \text{ in } F, x \text{ and } y \text{ in } A).$$

If $(e_1, \cdots, e_n)$ is a basis for A over F, then the ring-multiplicative structure of A is determined by the n^3 members c_{jki} of F (the "structure constants") defined by

$$e_j e_k = \sum_i c_{jki} e_i.$$

The *left regular representation* of A (with respect to this basis) is the scalar-multiplication-preserving homomorphism of A into $M_n(F)$ (the algebra of $n \times n$ matrices over F) in which the image L_j of e_j is given by $(L_j)_{ik}=c_{jki}$; in the analogous *right regular representation* the image R_k of e_k is given by $(R_k)_{ji}=c_{jki}$. A change of basis in A alters these homomorphisms only by a similarity transformation, so that for our purposes there is no loss of generality in working only with the fixed basis $(e_1, \cdots, e_n)$.

The best-known sufficient condition that both regular representations of A be *faithful* (*i.e.*, be isomorphisms into $M_n(F)$) is that A *have a unity*. A weaker sufficient condition, noted in ([1] p. 292, Ex. 5) is that *not every element of* A *be a zero-divisor*; more precisely, the left (right) regular representation of A is faithful if A has at least one element which is not a right (left) zero-divisor. (An element x of A is a *right zero-divisor* if $yx=0$ for some nonzero y in A; *left zero-divisor* is defined analogously.) This sufficient condition is not necessary (as we show by an example below), and so it is esthetically desirable, though mathematically rather trivial, to state explicitly the "natural" necessary and sufficient condition:

THEOREM. *The left (right) regular representation of A is faithful if and only if A has no nonzero left (right) "annihilators."*

Proof. The image $\sum_j c_j L_j$ of the element $x=\sum_j c_j e_j$ of A under the left regular representation can vanish if and only if

$$\sum_j c_j c_{jki} = 0 \qquad \text{(all, } k, i\text{)},$$

and thus if and only if

$$xe_k = \sum_i \left(\sum_j c_j c_{jki} \right) e_i = 0 \qquad \text{(all } k\text{)},$$

and thus if and only if $xy=0$ for all y in A; *i.e.*, if and only if x is a left "annihilator" of A. The proof of the "right-handed" version is similar.

To round out our discussion, we give an example (found by M. Newman) in which (a) every element of A is both a right zero-divisor and a left zero-divisor, but (b) no nonzero element of A is either a right annihilator or a left annihilator. For such an example, take A to consist of all bordered 3×3 matrices (with entries in F) of the form

$$M_1 = \begin{pmatrix} a_1 & 0 & 0 \\ b_1 & 0 & 0 \\ c_1 & d_1 & f_1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} a_2 & 0 & 0 \\ b_2 & 0 & 0 \\ c_2 & d_2 & f_2 \end{pmatrix},$$

so that

$$M_1 M_2 = \begin{pmatrix} a_1 a_2 & 0 & 0 \\ b_1 a_2 & 0 & 0 \\ c_1 a_2 + d_1 b_2 + f_1 c_2 & f_1 d_2 & f_1 f_2 \end{pmatrix}.$$

Each element M_1 of A is a left zero-divisor, since we can take $a_2=d_2=f_2=0$ and then choose nonzero b_2 and c_2 for which $d_1 b_2 + f_1 c_2 = 0$; similarly each element of A is a right zero-divisor. No nonzero element M_1 of A is a left annihilator, for if $M_1 M_2 = 0$ for all M_2 then we can take $a_2 \neq 0$ and $d_2 \neq 0$ to prove $a_1=b_1=f_1=0$, then take $a_2 \neq 0$ and $b_2=0$ to prove $c_1=0$, then take $b_2 \neq 0$ to prove $d_1=0$; similarly no nonzero element of A is a right annihilator.

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A THEOREM IN THE DECIMAL REPRESENTATION OF RATIONALS

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We present here a theorem concerning the number of times a given digit appears in the repeating sets one gets upon division by a prime. We will assume throughout the paper that m denotes the numerator and n or p denotes the denominator of a fraction and that $(10m, n)=1$, $m<n$. Let s be the number of digits in the smallest repeating set of m/n.

Let us first recall a few facts about decimals:

(1) The quantity s depends only on n, and $s(n)\,|\,\phi(n)$, where $\phi(n)$ is the Euler function.

(2) The $\phi(n)$ possible values for m give exactly $\phi(n)/s(n)$ repeating sets such that no two are cyclic permutations of each other. We call these the *distinct repeating sets.*

(3) As m varies, each digit in each distinct repeating set appears exactly once in "first place," that is, immediately after the decimal point.

The proof, which we do not give completely here, depends upon the fact that if $n\,|\,(10^t-1)$, then $s\,|\,t$, and vice versa; Euler's theorem tells us that $n\,|\,(10^{\phi(n)}-1)$. There are $\phi(n)$ repeating sets in all, because there are $\phi(n)$ possible values for m; there are exactly $s(n)$ cyclic permutations for each repeating set, since there are that many digits in each set. Hence there are $\phi(n)/s(n)$ distinct repeating sets.

THEOREM. *Consider the collection of distinct repeating sets of any prime. If $G(d_i)$ is the total number of times the digit d_i appears in these repeating sets, then for all i, j, $|G(d_i)-G(d_j)| \leqq 1$.*

This means simply that as far as possible, every digit appears just as often as any other digit.

Proof. We have already seen that every digit in each distinct repeating set appears exactly once in "first place" as we consider the repeating sets of m/p (p being prime) for all m less than p. And we get this first digit by solving the equation $10m=d_ip+r$, for $0<r<p$. The question now resolves itself into the following: How many m's exist for each d_i so that both the equation and the condition on r are satisfied?

In order for the condition on r to be satisfied, $d_ip<10m<(d_i+1)p$.

Also if $\bar{m}$ is the least such m, and $\bar{m}+k$ is the greatest such m, then $d_ip<10\bar{m} \leqq 10(\bar{m}+k)<(d_i+1)p$. Hence $d_ip+10k<(d_i+1)p$, and $p>10k$.

On the other hand, since $\bar{m}$ is the least such m, and $\bar{m}+k$ is the greatest, $10(\bar{m}+k+1)>(d_i+1)p$, and $10(\bar{m}-1)<d_ip$. Therefore, $10(\bar{m}+k+1)>(d_i+1)p>p+10(\bar{m}-1)$, and $10(k+2)>p$.

Finally, $p/10>k>p/10-2$. Since the total number of possible m's is $k+1$, then for any digit d_i, $p/10+1>G(d_i)>p/10-1$. From this we can see that $G(d_i)$ is one of two possible integers.

COROLLARY. *If we write p as $10q+r$, $(0<r<10)$, then $(11-r)$ digits appear q times, and $(r-1)$ digits appear $q+1$ times.*

Proof. If x digits appear k times, and $(10-x)$ digits appear $(k+1)$ times, then x and k must satisfy the equation $xk+(10-x)(k+1)=p-1$. It is easy to show that the solution to this is unique, and hence $x=11-r$ and $k=q$.

Let us call a digit which appears q times *deficient* and one which appears $(q+1)$ times *excessive*.

If we work with the inequalities $d_i(10q+r)<10m<(d_i+1)(10q+r)$, we see readily that if $r=1$, all digits are deficient;

if $r=3$, there are two excessive digits, 3 and 6;

if $r=7$, four digits are deficient: 0, 3, 6, and 9;

if $r=9$, two digits are deficient: 0 and 9.

For example, take $p=73$. There are nine distinct repeating sets, each with eight digits, as follows:

01369863, 02739726, 04109589, 05479452, 06849315, 08219178, 12328767, 16438356, 24657534.

In accordance with the corollary, the digits 3 and 6 appear eight times each, and every other digit appears seven times.

CLASSROOM NOTES

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ON EXTREMAL OPTICAL PATHS AND THE LAW OF REFLECTION

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Consider the following problem: Given a simple plane curve C having curvature at every point and given two points p_1 and p_2 both on the same side of C, determine that point p of C which minimizes $l=|\mathbf{p_1p}|+|\mathbf{pp_2}|$.

When C is a straight line, we have the famous problem of Heron (see [1], pp. 330–1 and [2], pp. 142–4). The solution to Heron's problem is that the lines $\mathbf{pp_1}$ and $\mathbf{pp_2}$ must make equal angles with the given line (see the above-mentioned references for a proof).

The above principle of Heron was generalized by Fermat to a principle of least time for optical ray tracing. In particular, for our general curve C any point p of C such that the path p_1pp_2 obeys the reflection law should minimize l.

Consider, however, the case where C is a circle and p_1 and p_2 are two distinct points on C. The points p which obey the reflection law are the endpoints of that diameter of C which bisects $\mathbf{p_1p_2}$. It is easily verified that one of these points maximizes l while the other yields at least a relative maximum of l.

That the principle of least time should be reformulated as a principle of extremal time has been realized by physicists (see [3], pp. 3–7). It would seem, however, from [1] and [2] that mathematicians are not generally aware of this. In the present note, we shall show that the extremal nature of a path of reflection depends on the value of the curvature at the point of reflection. While this result is known ([3], p. 5), our use of simple techniques from differential geometry should be of pedagogic value.

Let $\mathbf{r}$ be the radius vector from a fixed origin to the point p of C and let $\mathbf{r}_1$ and $\mathbf{r}_2$ be radii vectors from the same origin to p_1 and p_2. Let $\mathbf{r}$ be given as a function of arc length s. Then $l=|\mathbf{r}-\mathbf{r}_1|+|\mathbf{r}-\mathbf{r}_2|$.

Using the differentiation rule

$$\frac{d|\mathbf{v}|}{du}=\frac{\dfrac{d\mathbf{v}}{du}\cdot\mathbf{v}}{|\mathbf{v}|} \qquad (|\mathbf{v}|\neq 0),$$

we find that

$$\frac{dl}{ds}=\frac{(\mathbf{r}-\mathbf{r}_1)\cdot\dfrac{d\mathbf{r}}{ds}}{|\mathbf{r}-\mathbf{r}_1|}+\frac{(\mathbf{r}-\mathbf{r}_2)\cdot\dfrac{d\mathbf{r}}{ds}}{|\mathbf{r}-\mathbf{r}_2|} \tag{1}$$

provided $\mathbf{r}\neq\mathbf{r}_1, \mathbf{r}_2$ (as would be the case at a reflection point). But $d\mathbf{r}/ds$ is the unit vector $\mathbf{t}$ which is tangent to C at p and points in the direction of increasing s. Hence

$$\frac{dl}{ds}=\cos\alpha_1-\cos\alpha_2, \tag{2}$$

where α_1 and α_2 are as in Figure 1 (if the direction of $\mathbf{t}$ is opposite to that of Figure 1, α_1 and α_2 must be reversed in (2)).

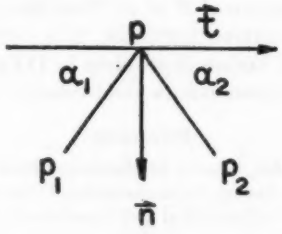

FIG. 1

Thus it is seen that the law of reflection holds at all points for which $dl/ds=0$. In particular, the law will hold at all points of C if and only if l is constant, hence if and only if C is an ellipse with foci at p_1 and p_2.

To investigate the nature of the reflection points as critical points of l, let us compute d^2l/ds^2. In the computation the following conventions are used: $\mathbf{n}$ denotes a unit vector normal to C at p, pointing toward the side of C on which p_1 and p_2 lie; K denotes the curvature of C at p, taken as positive when the center of curvature lies on the same side of C as p_1 and p_2 and negative when the center of curvature is on the opposite side. Thus $d^2\mathbf{r}/ds^2 = d\mathbf{t}/ds = K\mathbf{n}$.

Differentiating (1), we obtain

$$\frac{d^2l}{ds^2} = \frac{1 + (\mathbf{r} - \mathbf{r}_1)\cdot\dfrac{d^2\mathbf{r}}{ds^2} - \left[(\mathbf{r} - \mathbf{r}_1)\cdot\dfrac{d\mathbf{r}}{ds}\right]^2}{|\mathbf{r} - \mathbf{r}_1|^2} + \frac{1 + (\mathbf{r} - \mathbf{r}_2)\cdot\dfrac{d^2\mathbf{r}}{ds^2} - \left[(\mathbf{r} - \mathbf{r}_2)\cdot\dfrac{d\mathbf{r}}{ds}\right]^2}{|\mathbf{r} - \mathbf{r}_2|^2}$$

(3)

$$= \frac{1}{|\mathbf{r} - \mathbf{r}_1|^2} + \frac{1}{|\mathbf{r} - \mathbf{r}_2|^2} - \cos^2\alpha_1 - \cos^2\alpha_2 - K\left(\frac{\sin\alpha_1}{|\mathbf{r} - \mathbf{r}_1|} + \frac{\sin\alpha_2}{|\mathbf{r} - \mathbf{r}_2|}\right).$$

At a reflection point, $\alpha_1 = \alpha_2 \neq 0$, and we see that there is a quantity K_0 (which we shall soon show to be positive) such that

$$\frac{d^2l}{ds^2}\begin{cases} \text{is positive} & K < K_0, \\ \text{is zero} & K = K_0, \\ \text{is negative} & K > K_0. \end{cases}$$

Hence, at a reflection point, we see that l has a relative minimum if $K < K_0$ and a relative maximum if $K > K_0$. If $K = K_0$, further investigation is needed.

Now let us investigate K_0. We could, of course, compute it from (3). Such a computation would not tell us much. Note, however, that K_0 depends only on the quantities $\alpha_1 = \alpha_2$ and $|\mathbf{r} - \mathbf{r}_1|$ and $|\mathbf{r} - \mathbf{r}_2|$.

Consider now the ellipse E having foci at p_1 and p_2 and passing through the reflection point p. This ellipse, obeying the reflection law at all points, is necessarily tangent to C at p. Furthermore, on E, $l = \text{constant}$ and $d^2l/ds^2 = 0$. Hence we see that K_0 is the curvature of E at p. Thus K_0 is positive.

Thus whenever K is negative (example, C a circle, p_1 and p_2 lying outside of C as in the discussion of Steiner's problem in [1] and [2]), we are automatically guaranteed that reflection points yield relative minima in l.

References

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ON THE GREGORY INTERPOLATION FORMULA

C. T. LONG, Washington State University

A great deal has been said in recent years about the place of inductive reasoning in the teaching of mathematics. George Pólya, in particular, has maintained that if the teaching of mathematics is to reveal to any degree the invention of mathematics then an effort must be made to teach guessing or plausible reasoning. One particularly nice problem, which the author has found stimulating to students and quite suitable for illustrating the inductive approach, is the derivation of the well-known and useful Gregory interpolation formula.

Suppose, for example, one considers the sequence 2, 3, 3, 6, 16, 37, $\cdots$ and asks if a simple rule can be deduced by which successive terms might be formed. The students will readily suggest that the method of taking differences is often helpful in answering such questions and they may be led to guess that taking differences of differences or forming an array of differences may also be helpful. In the present case one would obtain the array

$$\begin{array}{ccccccc} 2 & & 3 & & 3 & & 6 & & 16 & & 37 \\ & 1 & & 0 & & 3 & & 10 & & 21 & \\ & & -1 & & 3 & & 7 & & 11 & & \\ & & & 4 & & 4 & & 4 & & & \end{array} \tag{1}$$

which finally ends with a row of constant differences. In this case it is clear that the third row of (1) is simply an arithmetic progression whose terms are generated by the formula

$$h(s) = -1 + 4s, \tag{2}$$

if the first term is assumed to correspond to $s=0$, the second to $s=1$, and so on.

Now the elements of the second row of (1) are easily obtained by adding to 1 the appropriate terms from the third row. That is, the elements of the second row are generated by the formula

$$\begin{aligned} g(t) &= 1 + \sum_{s=0}^{t-1} h(s) \\ &= 1 + \sum_{s=0}^{t-1} (-1 + 4s) \\ &= 1 - t + 4 \cdot \frac{t(t-1)}{2}, \end{aligned} \tag{3}$$

where again the first term is assumed to correspond to $t=0$, the second to $t=1$, and so on.

At this point one is tempted to simplify (3) immediately to $g(t) = 1 - 3t + 2t^2$ but a more careful look at (1), (2), and (3) makes the correspondence of the

coefficients 1, -1, and 4 seem somewhat more than just coincidental and this correspondence would be obscured if the simplification were made. One might even begin to suspect that a general pattern is beginning to emerge for the form of the generating function for a sequence whose array of differences terminates in a row of constant differences. Of course, ideas of this sort might be tested by guessing the generating function for the given sequence and then deriving the correct formula as we did in (2) and (3) above. Looking again at (1) we conjecture that the generating function for the given sequence ought to be

$$f(n) = \frac{2}{0!} + \frac{n}{1!} - \frac{n(n-1)}{2!} + \frac{4n(n-1)(n-2)}{3!}. \tag{4}$$

There may, of course, be some trouble in guessing the divisor for the last term. Reasonable candidates for this divisor might be 2^3, 3, or 3!, but students familiar with the binomial theorem usually choose 3! and modify the divisors on the first three terms as shown.

Now the nth term of the given sequence is clearly equal to 2 plus the sum of the first $n-1$ terms of the second row of (1). Thus the correct formula for the generating function of the given sequence is given by

$$\begin{aligned} &2 + \sum_{t=0}^{n-1}\left[1 - t + 4\cdot\frac{t(t-1)}{2}\right] \\ &= 2 + n - \frac{n(n-1)}{2} + 4\sum_{t=0}^{n-1}\frac{t(t-1)}{2} \\ &= 2 + n - \frac{n(n-1)}{2} + 4\sum_{t=0}^{n-1}\left[\frac{(t+1)t(t-1)}{3!} - \frac{t(t-1)(t-2)}{3!}\right] \\ &= \frac{2}{0!} + \frac{n}{1!} - \frac{n(n-1)}{2!} + \frac{4n(n-1)(n-2)}{3!}, \end{aligned} \tag{5}$$

which is precisely the result already guessed in (4). It should also be pointed out that the educated guess in (4) helped us to evaluate the last sum in (5) by suggesting how to convert it into a collapsing sum.

Now let a_0 be the first term of a sequence of terms whose mth row of differences are constant and whose terms correspond to $n=0, 1, 2, \cdots$, and let a_i denote the first term in the ith row of differences for $i=1, \cdots, m$. The students will now readily guess that the generating function for the given sequence should be

$$F(n) = \sum_{i=0}^{m} a_i \binom{n}{i}. \tag{6}$$

Of course, if the terms of the given sequence correspond to $n=k, k+1, \cdots$ for some positive integer k, then $n-k$ should be substituted for n on the right side of (6).

Finally, the interpolation formula can now be proved by mathematical induction if we first prove by induction or by converting to a collapsing sum the identity

$$(7)\qquad \sum_{i=0}^{n-1} \frac{i(i-1)\cdots(i-i)}{(i+1)!} = \frac{n(n-1)\cdots(n-i-1)}{(i+2)!}, \qquad 0 \leqq i \leqq n-2,$$

which is also suggested by the preceding discussion.

A NOTE ON THE FUNCTION $W=A_z+B$

E. S. ELYASH and NORMAN LEVINE, University of Pittsburgh

A complex function $w=f(z)$ will be said to satisfy condition L if and only if $|z_1-z_3| = |z_1-z_2| + |z_2-z_3|$ always implies that $|f(z_1)-f(z_3)| = |f(z_1)-f(z_2)| + |f(z_2)-f(z_3)|$. It is well known that the function $w=Az+B$, $A\neq 0$, satisfies condition L and is conformal; it is the purpose of this note to give a proof of the converse. To the best of the authors' knowledge this converse does not appear in the above formulation elsewhere.

THEOREM. *If $w=f(z)$ is conformal and satisfies condition L, then $w=Az+B$, where A and B are constants and $A\neq 0$.*

Proof. Let z_0 be a fixed point and R a ray from z_0 with $\arg(z-z_0)=\theta$ for all $z\in R$. Since R is connected and $f(z)$ is continuous, the image $f(R)$ is also connected. Furthermore $f(z)$ being conformal, $f(R)$ is nondegenerate and, as an obvious result of condition L, $f(R)$ is a ray or a line-segment. Hence let $\arg(w-w_0)=\phi$ for all $w\neq w_0$ and $w\in f(R)$; thus an argument for $f'(z_0)$ is $\phi-\theta$. By application of the same reasoning to any other point on R, $\arg f'(z)=\arg f'(z_0)$ for all $z\in R$. Indeed, by considering any other ray through z_0 and using the analyticity of $f(z)$ and $f'(z)$ we have $\arg f'(z)=\arg f'(z_0)$ for all z. This permits us to say

$$f'(z) = \rho(x, y)e^{i(\phi-\theta)} \qquad \text{for all } z.$$

The Cauchy-Riemann conditions yield the pair of equations

$$\rho_x \cos(\phi-\theta) = \rho_y \sin(\phi-\theta),$$
$$\rho_y \cos(\phi-\theta) = -\rho_x \sin(\phi-\theta).$$

The unique solution of this system is $\rho_x=\rho_y=0$ and therefore ρ is a constant. Thus $f'(z)=A$ where A is a complex constant different from zero because $f(z)$ is conformal and finally $f(z)=Az+B$.

SIMPLE PROOF OF A FUNDAMENTAL THEOREM OF FIELD THEORY

A. KERTÉSZ, University of Debrecen, Hungary

Let L be an extension of the field K and let A and B be two subsets of L. We say that A algebraically depends on B (over K) if each element of A is algebraic over $K(B)$. A and B are called algebraically equivalent (over K) if A depends on B and B on A. The set A is algebraically independent (over K) if no element x of A depends algebraically on the set $A-x$.

In the theory of fields the following theorem of Steinitz is of fundamental importance.*

THEOREM. *Let A and B be two subsets of L, algebraically independent and equivalent (over K). Then A and B have the same cardinal number.*

It is the aim of this note to give a simple proof of this theorem, which makes use only of the well-known set-theoretical fact that for an arbitrary infinite cardinal $\mathfrak{p}$ one has $\aleph_0\mathfrak{p}=\mathfrak{p}$ (which relation is based on the well-ordering theorem) and of the following corollary of the so-called exchange theorem of Steinitz: If the algebraically independent finite set N of n elements algebraically depends on the set M, then M has at least n elements.†

Let us indeed prove the above theorem. If one of the sets A and B is finite, then by virtue of the mentioned corollary of the exchange theorem A and B have the same cardinal number. Let us suppose, therefore, that A has cardinal number $\mathfrak{m}$, B has cardinal number $\mathfrak{n}$, with both cardinals infinite and $\mathfrak{m}>\mathfrak{n}$. To each element of A we make correspond one (and only one) finite subset of B, on which this element of A algebraically depends. Thus we map the set A into the set S of all finite subsets of B. If the complete inverse image of each element of S occurring in the range of this mapping were finite, then the cardinal number $\mathfrak{m}$ of A must be such that $\mathfrak{m}\leqq\mathfrak{n}$ since the cardinal number of S itself is at most $\aleph_0\mathfrak{n}=\mathfrak{n}$. So, in view of our hypothesis that $\mathfrak{m}>\mathfrak{n}$, the set A must have an infinite subset which algebraically depends on the same finite subset of B. This, however, contradicts the mentioned corollary of the exchange theorem. The reason for this contradiction is clearly the hypothesis that $\mathfrak{m}>\mathfrak{n}$, and in view of the fact that the rôle of A and B is completely symmetric, it follows that $\mathfrak{m}=\mathfrak{n}$, thus completing the proof.

* E. Steinitz, Algebraische Theorie der Körper (edited by R. Baer and H. Hasse), Berlin, 1930, New York, 1950, Section 23.

† *Ibid.*, Section 22, Theorem 9. For a simpler proof see G. Pickert, Einführung in die höhere Algebra, Göttingen 1951; pp. 65–67, 172–173.

MATHEMATICAL EDUCATION NOTES

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THE EDUCATION OF MATHEMATICS TEACHERS*

Introduction

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The topic to be discussed by our panelists is an important one. The improvement of instruction in mathematics in the secondary school or of the secondary school curriculum depends in the last analysis on the teacher. He holds the key. Only the teacher can translate recommendations into classroom activity.

But such improvement requires more than good will on the part of the teacher. It requires a better knowledge of subject matter. The improvement of secondary school mathematics does not require improved methodology; it requires enhanced knowledge of mathematics on the part of secondary school teachers.

Moreover, it requires that the teacher continue to learn throughout his professional career. We must have done with the notion that a teacher's education is complete when he receives his Bachelor's degree.

The problem of teacher education is twofold. There is the problem of preservice education: what mathematics should be taught to the undergraduate who is preparing to be a secondary school teacher of mathematics? There is also the problem of in-service education: what mathematics should be taught to the teacher who completed his formal education some time in the past, but who needs to bring his knowledge of mathematics up to date in order to modernize the instruction given in his classroom? It is hoped that our panelists will make remarks on both aspects of the problem of teacher education.

Finally, let me say that this particular panel will almost certainly present proposals that in some measure at least will reflect the recommendations of the Commission on Mathematics of the College Entrance Examination Board, for its members have all had some contact with its work. The basic objective of the program recommended by the Commission for college-capable high school students is that they should be prepared to begin a course in calculus in their freshman year in college. Stress is therefore placed on algebra and on the study of the properties of the elementary functions, with a somewhat lessened emphasis (at least in terms of time allotted to the subject) upon geometry. Skills are regarded as very important, but the development of concepts, of an understand-

* Edited transcriptions of talks given as part of a panel discussion at the annual meeting of the New Jersey Section of the Mathematical Association of America on November 1, 1958. Several others of the panel talks will appear in a later issue.

ing of the nature of deductive reasoning, and of mathematical insight, power and understanding are of even greater importance.

The attainment of such high objectives demands thoroughly prepared teachers. What this preparation should be is the theme of the panel discussion.

Algebra

E. R. Lorch, Columbia University

In a recent discussion with Jean Vilar, the producer of the French National Theater, the question arose as to whether he had to choose special plays to get down to the level of his auditors. In reply he said that he had discovered that some of the most intricate, deepest plays are the ones which are best received by unsophisticated groups. And his counsel was, "What the audience wants is poetry."

Now I think that this is an answer that we could paraphrase for our mathematical work. What the student wants is mathematics, mathematics of the deepest, of the most genuine, of the purest type. Let us give him that, no matter if it now seems a little bit abstruse, or if it should seem to him a little bit difficult. Possibly we should go at it in gentle steps, but no matter what the difficulties are, we should go straight to the core of the subject matter and base our program for the undergraduate on genuine mathematics.

It so happens that mathematics in general has been undergoing a remarkable period of cleaning up. While the nineteenth century was a century in which mathematics was made rigorous—take, for instance, the structure of the real number system—this is the century in which mathematics is being made clean; and possibly algebra is that branch of mathematics in which the cleanliness is most apparent. That being the case, I suggest that we go in for algebra in big doses.

As specific suggestions, I have a list of five main categories. The first concerns the *Algebra of Sets and Theory of Sets in General.* My first subheading is something which everyone would think of at once in connection with algebra of sets, namely, the algebra of complementation, of union and intersection, the theory of duality, and the relation of the calculus of sets with the Boolean algebra of propositions. This material is very standard, and something with which we are all familiar. This is what most people mean, I think, when they speak of introducing set theory into the curriculum.

The next subheading under algebra of sets I consider to be less important. I mention it only for the sake of completeness: the arithmetic of cardinals and ordinals. Since the work of Cantor, this has been considered classical material of great interest; books on the theory of sets have been dedicated virtually to this subject alone. Yet I do not think it particularly important. The relation between the two smallest cardinals, C and aleph-null, should be handled at the college level, but not the general arithmetic.

Going ahead with our subheadings, we should talk about the introduction of certain other set-theoretic operations involving constructions of various types,

starting with given sets. For instance, an obvious set to construct if a set E is given, is the set of all subsets of E. Another very important operation given two sets E and F, is to construct their product, $E \times F$. Indeed, this product is one of the fundamental bricks in the mathematical structure. From the product $E \times F$ one can go on to the definition of relations involving E and F. There are roughly three special types of relations which should be mentioned: the equivalence relation and the related partition theorem that an equivalence relation generates a partition of a certain type in the set in which it operates; functional relations, which lead to the definition of a function and hence are obviously of the utmost importance in all undergraduate work; and finally, order relations: partial ordering, total ordering, and well-ordering. All of these can be illustrated copiously by material which is readily at hand in the general undergraduate curriculum.

So much for the algebra of sets. Heading II is *The Theory of Groups and Rings*. I would start with a discussion of the role of axiomatics in modern mathematics as distinguished from its very limited role of fifty years ago. This modern spirit may be illustrated by considering groups and rings. Of the utmost interest to the high school teacher is first of all the meaning of the symbol $-a$ as applied to a group (in general the meaning of subtraction). Based on this operation are two little things that will have to be proved for the students. One is that $-(-a)$ is equal to a and the other is that $-a$ times $-b$ is equal to ab. These are two little gems in the theory of groups and rings and their proof constitutes an introduction to abstract algebra. The definition of a field should be given, followed by a discussion of the problem of imbedding a domain of integrity (a certain kind of a ring) in a field. For example, starting with the integers one thus obtains the rational numbers. This theorem is a very beautiful result and it shows the student precisely what the rationals are.

Next I put down two other topics: the elements of the theory of ideals, including definitions of ideal, and prime ideal, which can be very simply given, once more with illustrations. In that connection it should be shown how the complex numbers are constructed from the real numbers by certain prime ideal operations. This is a construction which is a little better, I consider, than the classic Cauchy method of pairs, in which multiplication and addition are defined in what seems to be an arbitrary way.

Let me go through the other three headings rapidly. III: *Rings in Analysis*. There are special rings in analysis which require attention. Rings of functions—the functions which one integrates or differentiates form a ring: add them or multiply them, one still has integrable or differentiable functions. And this leads to a notion which might be mentioned sooner or later: that around the corner lurking somewhere is a structure known as a Banach space and even a more complicated one called a Banach algebra. Specifically one could talk about the space or the ring of the continuous functions on the closed interval $[a, b.]$

IV: *Number Theory*. We begin with the theory of ordinary integers. Out of this flows the theory of finite fields, especially the prime fields with p elements. They can be studied in terms of the elementary congruences of number theory.

And possibly something can be done in algebraic number theory to show that factorization of the primes is not necessarily unique, that it is only unique in certain types of situations. Simple examples can be given to illustrate non-uniqueness, remedied once more with the theory of ideals.

This brings me to section V, where I should like to discuss *Linear Transformations* and possibly, for those who would like me to do it in the old-fashioned way (although some of my colleagues would shudder at it), plain ordinary matrices. In that connection beautiful illustrations of the notions of nilpotence, idempotence, and noncommutativity are available.

This, to my mind, constitutes a reasonable program of algebra for undergraduates.

Analysis

A. W. TUCKER, Princeton University

It is, of course, clear that the basic part of analysis for the training of a secondary school teacher is calculus. I would like to see courses in calculus improved in various directions, but these directions are not novel. The ideal calculus book in spirit, I feel, is Courant's *Calculus*. Unfortunately, it is not in a form that is easy for an undergraduate to read. It is written more in the style of what we regard as a graduate textbook but the compromise, the wonderful compromise, that it embodies between intuition, applications and careful mathematics is ideal. We have other books in calculus that are very fine in certain directions. There has been a tendency recently to go very heavily in the direction of mathematical rigor. I think that this care on the rigorous side is more appropriate in algebra than in calculus. It is the life blood of the calculus that intuition have full use and that there be applications of all possible sorts—from physics, of course, from engineering, but also the social sciences (marginal concepts from economics, for example). The great value of the calculus is the bringing together of intuition and the tremendous wealth of applications as well as the serious mathematical core that is regarded as the backbone of college mathematics.

The importance for the secondary school teacher of calculus is to shed light on the elementary functions—the polynomial, exponential and logarithmic, and circular functions—and, therefore, the calculus training of the prospective teacher should go into the differential equations that pertain to the exponential and circular functions. Also, since complex numbers should be an essential part of the secondary school mathematics, some introduction to functions of a complex variable would be desirable for a secondary school teacher. So calculus, as you have it in Volume I of Courant's *Differential and Integral Calculus*, should include the differential equations I referred to and an introduction to functions of a complex variable. These I would regard as the desiderata for the analysis training of a prospective secondary school teacher.

If you ask me to go on and mention other possible topics in analysis I could do so but I feel that analysis now does not play quite the dominant role that

it has in the past. Calculus is still probably the most important connection between school mathematics and advanced mathematics, but parallel to the calculus we have algebra, especially, and also probability, so we have to think of mathematical education no longer as a single track but a number of parallel tracks with much switching back and forth between the tracks.

THE BIOLOGICAL SCIENCES CURRICULUM STUDY

ARNOLD B. GROBMAN, University of Colorado

After an initial grant was made to the American Institute of Biological Sciences by the National Science Foundation, the Biological Sciences Curriculum Study was established with Dr. Bentley Glass, Johns Hopkins University, as Chairman of its Steering Committee and with the writer as Director of the Study. Late in January 1959, headquarters for the BSCS were established at the University of Colorado. A newsletter is available upon request.

Several projects were organized at the first meeting of the Steering Committee in early February; these, of course, are but a sample of the investigations that will eventually be undertaken.

Course Content in the Biological Sciences. Probably the central activity will be the work of a special committee that will attempt to answer the general question: What should a person graduating from high school know about the biological sciences? At a later time, studies of university and college curricula will be undertaken.

Innovation in Laboratory Instruction. The BSCS group felt rather strongly that much laboratory instruction in biology is less than inspiring and that a great deal could be done to make laboratory experiences more educationally meaningful. It is hoped that some pilot experiments utilizing some of the laboratory exercises designed by the committee will be attempted in the fall of 1959.

Studies of Successful Biology Teachers. There are a considerable number of high school biology teachers who are generally recognized in their communities as exceptionally effective teachers. The BSCS is interested in assembling information about a representative group of these teachers so that their backgrounds, training, and methods can be analyzed.

Digest of Existing Information. There is a wealth of information available that has been assembled by federal and state agencies, scientific and educational societies, and various other organizations. The BSCS will attempt to evaluate this mass of information and to prepare a digest.

Reviews for Teachers in the Life Sciences. As an aid to teachers now in service, to students preparing for teaching careers who may be unable to elect a full program in the biological sciences, and to laymen with an interest in the general area, the BSCS plans to initiate a pamphlet series. Each pamphlet would be concerned with a specific topic, would be written by an outstanding authority, and would be carefully designed for its appeal to the intended audience.

PROGRAMS IN MATHEMATICS OF ACADEMIES OF SCIENCE, 1959–60

Fifty proposals from 30 regional scientific organizations, usually State Academies of Science, have been approved by the National Science Foundation for participation in the new NSF Academies of Science Program.

A letter was sent by J. R. Mayor to the director of each of these activities requesting information on the place of mathematics in all of these programs. Replies from nine of the 30 directors indicate that their programs will include mathematics as one of the sciences. At least two of the proposals are for mathematics alone.

Brief descriptions of the place of mathematics in these nine programs are given below. According to the outline submitted by the National Science Foundation, it appears that at least four or five of the others, from which no response has been received, will include mathematics in the science education activities.

The *Buffalo Society of Natural Sciences* is planning to include mathematics as a part of its Project Stimulation program this winter, but details are not available. It possibly will take the form of projects for presentation at the Science Congress (science fair) as exhibits or lecture demonstrations. It expects to use visiting lecturers of the university level for this work.

The *Chicago Academy of Sciences* indicated that it would welcome lectures on mathematics (it has had a couple) that show the bearing of mathematics on astronomy, earth movements, biology, and any other similar applications. It would also like to cooperate with the State section of the Mathematical Association of America in any way feasible. It will seek ways in which mathematics can be applied to the several study courses upon which the Academy is embarked.

The *Cooper Union for the Advancement of Science and Art* is sponsoring a Mathematics Speakers Bureau in metropolitan New York. Mathematicians will speak in high schools and participate in discussions involving students and teachers. It is expected that 235 schools will be served.

One of the projects of the *Hawaiian Academy of Science* which was approved by the NSF is that of Teachers' Science Seminar Series which involves sending distinguished scientists out to give lectures designed to give teachers subject-matter information. These illustrated lectures also are open to the public in some cases and the local department of instruction allows partial credit for teachers who participate in a series. So far there have been no lectures in the field of mathematics, but there is a chance that one or two lectures in mathematics might be sponsored during the coming year. This will be a matter for the Teachers' Science Seminar Committee to determine.

The *Indiana Academy of Science* includes mathematics among its twelve divisions. It expects to include all of the divisional areas of science and mathematics in the visiting scientists program if personnel can be found to agree to cooperate. There probably are a number of mathematicians who will be interested since they have been active in the affairs of the Indiana Academy.

The *Louisiana Academy of Sciences* project includes Visiting Lecturers, and

$1500 has been allocated to this phase. It expects to call on local scientists to participate in the program and believes that they will do so because many in educational institutions, professional groups, and industry have volunteered their services in the past. Thus far, there has been no time to compile a list of those who would participate on a state-wide basis, but such a list will be made. It is anticipated that many will not ask for even so much as remuneration for their expenses, but this will be offered. Costs for approximately 100 speakers can be provided with the $1500. Any assistance that comes from scientific organizations such as the Mathematical Association of America will be appreciated and a request will be sent to the MAA for a list of mathematicians in Louisiana who would be willing to take part in the project. Full cooperation with the MAA Committee on Visiting Lecturers to Secondary Schools was offered.

The *Minnesota Academy of Science* has a mathematics section which will be headed during 1959–60 by David R. Lewis, 3167 Lake Johanna Boulevard, St. Paul 12. Mr. Lewis is employed by the Remington Rand-Univac Division of the Sperry Rand Corporation. A committee is being set up to administer the visiting scientist program and to coordinate the programs of the professional societies and other groups. The available talent will be spread among a large number of schools. The committee will include local representatives of the professional societies, which should provide for the necessary exchange of information.

The *Nebraska Academy of Sciences* has a section of the Mathematical Association of America and a section of the National Council of Teachers of Mathematics. Each of these retains its identity but meets with the Academy in its spring meeting. Visiting scientists will include mathematicians who undoubtedly will be members of these groups, and consultations with the mathematicians as individuals will be held. The cooperation and good counsel of the MAA Committee on Visiting Lecturers to Secondary Schools will be welcomed.

The *Oklahoma Academy of Science* project does not involve visiting lectureships, nor does it involve in-service training. The "3-R's" Conference sponsored by the Frontiers of Science Foundation last November attracted considerable interest in science and mathematics education in this state. As a result, some 125 community-level organizations, such as Civic Clubs or Chambers of Commerce, asked for consultative services on how science and mathematics programs may be improved. The Oklahoma project is to furnish these consultative services through the scientists and mathematicians from the various four-year colleges in the state. While the cooperation of the Oklahoma section of the MAA has not been solicited, use will be made of mathematicians as consultants.

There is a Mathematics Section of the *Tennessee Academy of Science*. The Tennessee program for next year is designed to help science and mathematics teachers. It will be organized under the three divisions of Biological Science, Physical Science, and Mathematics. During each quarter of the year—fall, winter, spring—an all-day institute program is planned. Each such program will have three sections according to the above categories.

The *Texas Academy of Science* has a Mathematics Section; or more exactly,

the Texas Section of MAA is affiliated with the Texas Academy of Science and has its winter meeting with the annual meeting of the Academy. Its project is carried out within the Academy with the full cooperation of the Texas Section, MAA. All the members of the steering committee are very active members of MAA. The Texas Academy will be glad to cooperate with the MAA Committee on Visiting Lecturers to Secondary Schools—the lecturers will be mathematicians.

The *Virginia Academy of Science* has a section of mathematics, astronomy, and physics. The mathematicians have been talking about forming a section of their own but this has not yet been done. The committee which will pick the professors to invite as visiting lecturers has on it a mathematician, a biologist, an experimental psychologist, a physicist, a chemist, a geologist, and a biochemist interested in medical science. Every college has been asked to appoint a representative who will work with this committee. These representatives are requested to send names of individuals whom they would like to visit their college as a visiting scientist. Some of them no doubt will ask for a mathematician. The committee would like to consider cooperation with the MAA Committee on Visiting Lecturers to Secondary Schools.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1386. *Proposed by Monte Dernham, San Francisco, California*

In overtaking a freight, a passenger train which is x times as fast takes x times as long to pass it as it takes the two trains to pass when going in opposite directions. Find x.

E 1387. *Proposed by L. T. Van Tassel, San Diego, California*

Find p such that the line $y=p$ divides the area of an arch of the sine curve $y=\sin x$ into parts in a given ratio to one another.

E 1388. *Proposed by H. W. Gould, West Virginia University*

Determine the largest integral value of n such that there exists a solution to the system of inequalities

$$k < x^k < k + 1, \qquad k = 1, \cdots, n.$$

Would it make any difference if $<$ were replaced by $\leqq$?

E 1389. *Proposed by E. M. Scheuer, Space Technology Laboratories, Los Angeles, California*

Evaluate the $n \times n$ determinant A_n whose (i, j)th entry is $a^{|i-j|}$.

E 1390. *Proposed by J. Łoś and J. Mycielski, Wroclaw, Poland*

A *decomposition* of a rectangle (rectangular parallelepiped) R is a dissection of R into smaller rectangles (rectangular parallelepipeds) by lines (planes) parallel to the sides (faces) of R. If a decomposition of R is not a refinement of some other decomposition of R, then the decomposition is said to be *primitive*.

(1) Show that there exist primitive decompositions of a rectangle into any number $n > 1$ of rectangles, except $n = 3, 4, 6$.

(2) Show that there exist primitive decompositions of a rectangular parallelepiped into any number $n > 1$ of rectangular parallelepipeds, except $n = 3, 4$.

SOLUTIONS

A Criterion for a Right Triangle

E 1356 [1959, 233]. *Proposed by F. Leuenberger, Zuoz, Switzerland*

Let I, O, r, R, K denote the incenter, circumcenter, inradius, circumradius, and area of a triangle T. Show that $(IO)^2 + K = r^2 + R^2$ if and only if T is a right triangle.

Solution by Leon Bankoff, Los Angeles, Calif. In any triangle, $K = rs$, where s is the semiperimeter. Now $s = r + 2R$ if and only if T is a right triangle, in which case

$$K = r(r + 2R) = r^2 + R^2 - (R^2 - 2Rr) = r^2 + R^2 - (IO)^2.$$

Also solved by A. N. Aheart, Shlomo Ben-Adam, L. D. Goldstone, R. H. Hou, J. P. Hoyt D. C. B. Marsh, C. S. Ogilvy, N. R. Riesenberg, L. A. Ringenberg, Chih-yi Wang, Dale Woods, Roscoe Woods, and the proposer. Late solution by Kjeld Ejrnaes.

Marsh and Roscoe Woods showed that for any triangle ABC

$$(IO)^2 + K = R^2 + r^2 + 4rR\sqrt{2} \sin (45° - A/2) \sin (45° - B/2) \sin (45° - C/2).$$

An Equation with a Unique Solution

E 1357 [1959, 233]. *Proposed by Albert Wilansky, Lehigh University*

Prove that for every number a the equation $x = -a + \sqrt{2} \sin [(a - x)/\sqrt{2}]$ has a unique solution.

I. *Solution by A. R. Hyde, West Hartford, Conn.* The solution is found by

the intersection of the line $y=(a+x)/\sqrt{2}$ with the curve $y=\sin[(a-x)/\sqrt{2}]$. Since the slope of the line is the same as the maximum slope of the sine curve, there must be exactly one intersection.

II. *Solution by D. C. B. Marsh, Colorado School of Mines.* For arbitrary but fixed a, the function

$$f(x) \equiv x + a - \sqrt{2}\sin[(a-x)/\sqrt{2}]$$

has slope $1+\cos[(a-x)/\sqrt{2}]$, which is nonnegative for all x. Thus $f(x)$ is monotone and, ranging over all real values, takes on the value zero precisely once.

Also solved by E. F. Allen, Shlomo Ben-Adam, D. A. Breault, J. L. Brown, Jr., R. F. Brown and Joel Levy (jointly), C. H. Cunkle, S. J. Einhorn, George Glauberman, Michael Goldberg, L. D. Goldstone, A. G. Grace, Jr., Alfred Gray, R. Holt, Jack Kohne and Richard Rizzo and Robert Whitley (jointly), Viktors Linis, C. S. Ogilvy, F. D. Parker, J. L. Pietenpol, S. D. Pratico, B. E. Rhoades, L. A. Ringenberg, Charles Sampson, W. A. Veech, Dale Woods, and the proposer.

The Roots of a Special Cubic

E 1358 [1959, 234]. *Proposed by N. R. Riesenberg, Brooklyn College*

Show that the roots of the cubic equation $64x^3-192x^2-60x-1=0$ are $\cos^2(2\pi/7)\sec(6\pi/7)$, $\cos^2(4\pi/7)\sec(2\pi/7)$, $\cos^2(6\pi/7)\sec(4\pi/7)$.

Solution by D. C. B. Marsh, Colorado School of Mines. We transform known results into the desired form as follows: $\cos(2k\pi/7)$, $k=0, 1, \cdots, 6$, are the roots of $\mathrm{Re}(w^7-1)=0$, where $w=u+iv$. The latter equation may be rewritten as

$$(u-1)(8u^3+4u^2-4u-1)^2=0.$$

Thus $\cos(2k\pi/7)$, $k=1, 2, 3$, are roots of

$$8u^3+4u^2-4u-1=0,$$

and $\cos^2(2k\pi/7)$, $k=1, 2, 3$, are roots of

$$64t^3-80t^2+24t-1=0.$$

Setting $t=3x/(4x-1)$, the above equation transforms into

$$64x^3-192x^2-60x-1=0,$$

whose roots are

$$x = t/(4t-3) = \cos^2(2k\pi/7)/[4\cos^2(2k\pi/7)-3]$$
$$= \cos^2(2k\pi/7)/\cos 3(2k\pi/7) = \cos^2(2k\pi/7)\sec(6k\pi/7),$$

$k=1, 2, 3$, as we wished to show.

Also solved by Leon Bankoff, Shlomo Ben-Adam, J. F. Bittner, P. L. Chessin, F. J. Duarte, Michael Goldberg, J. M. Kingston, Albert Nijenhuis, R. E. Shafer, and the proposer. Late solution by Kjeld Ejrnaes.

Bankoff pointed out that the problem is given by Rev. E. M. Radford, *Mathematical Problem Papers*, Cambridge (1923), p. 421.

A Set of Eight Positive Integers

E 1359 [1959, 234]. *Proposed by Leo and William Moser, Universities of Alberta and Saskatchewan*

(1) Given eight positive integers $a_1<a_2<\cdots<a_8\leqq 16$. Prove that there exists a k such that $a_i-a_j=k$ has at least three solutions.

(2) Find a set $a_1, \cdots, a_8$ for which $a_i-a_j=k$ has at most three solutions for any k.

Solution by J. L. Pietenpol, Columbia University. (1) If no three of the seven positive numbers a_2-a_1, a_3-a_2, $\cdots$, a_8-a_7 are to be equal, then their sum must be at least equal to $1+1+2+2+3+3+4=16$. But the sum is $a_8-a_1\leqq 15$. We have thus established the stronger result that $a_i-a_{i-1}=k$ has at least three solutions for some k.

(2) There are many examples, such as 1, 2, 3, 4, 7, 9, 12, 16.

Also solved by D. A. Breault, R. F. Brown and Joel Levy (jointly), G. W. Day, N. J. Fine, George Glauberman, Michael Goldberg, J. H. Hodges, A. S. Howard, D. C. B. Marsh, R. A. Melter, D. J. Persico, D. R. Wilder, and the proposers.

k-commutative Elements of a Group

E 1360 [1959, 234]. *Proposed by J. L. Brenner, Stanford Research Institute*

Call two elements a, b of a group "k-commutative" (B. Friedman) if every product of k factors, each factor being a or b, commutes with every other such product. Show that for every two elements a, b of a group, the set of all k for which a, b are k-commutative is an ideal in the set of nonnegative integers.

Solution by J. Hooley, Derby and District College of Technology, Derby, England. With a definition of 0-commutativity analogous to $a^0=b^0=e$, the unit of the group, G, every pair a, b in G is 0-commutative.

Let J be the set of nonnegative integers and, for given a, b in G, let I be the subset of J for whose members k we have a, b a k-commutative pair. Let n be a positive integer and let g and g' be products of nk factors, each factor a or b. By the associative law, gg' can be expressed as

$$g_1g_2\cdots g_ng_1'g_2'\cdots g_n',$$

where g_j, g_j' $(1\leqq j\leqq n)$ are products of k factors. Then, using associativity and n^2 applications of k-commutativity (k in I), we see that

$$gg'=g_1'\cdots g_n'g_1\cdots g_n=g'g,$$

and a, b are (nk)-commutative.

If a, b is a k-commutative and an $(i+k)$-commutative pair, and if h, h' are any products of i factors (a or b), then

$$hh'=a^{-k}a^khh'a^ka^{-k}=a^{-k}h'a^{2k}ha^{-k},$$

since a, b are $(i+k)$-commutative. Let p be the least member of J such that

$$p + i \equiv 0 \pmod{k}.$$

Then $p \leqq k$ and

$$hh' = a^{-k}h'a^{p}a^{k}a^{k-p}ha^{-k} = a^{-k}a^{k}h'a^{p+k-p}ha^{-k}$$

(using $(p+i)/k$ applications of k-commutativity)

$$= h'a^{k}ha^{p}a^{-p-k} = h'ha^{p}a^{k}a^{-p-k}$$

(again using $(p+i)/k$ applications of k-commutativity) $= h'h$. That is, if k and $k+i$ are in I, then i is in I.

It now follows that I is an ideal in J.

Also solved by James Baugh and T. G. McLaughlin (jointly), D. C. B. Marsh, D. J. Persico, and the proposer.

Baugh and McLaughlin established the stronger result that the ideal in the set J of nonnegative integers is actually a principal ideal in J.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4869. *Proposed by M. S. Klamkin, AVCO Research, Wilmington, Mass.*

A smooth centro-symmetric curve has the property that the centroid of any half-area which is formed by chords through the center is equidistant from the center. Show that the curve is a circle.

4870. *Proposed by D. S. Mitrinovitch, Belgrade, Yugoslavia*

If the variable z traces the circle $|z| = 1$, find and describe the path of the point $w = (z^2 - az)/(az - 1)$, where the parameter a is a complex constant.

4871. *Proposed by Juris Hartmanis, Ohio State University*

Prove that the Boolean algebra of all subsets of the integers has maximal chains whose length is denumerable and has also maximal chains whose length is not denumerable.

4872. *Proposed by Ky Fan, Oak Ridge National Laboratory*

Let A be a matrix (not necessarily square) of rank $r \geqq 1$ and with non-negative elements. Let A^* denote the transpose of A. Prove that the square matrix AA^* has no eigenvalue different from 0, 1, if and only if, after deleting all identically vanishing rows and columns, the remaining submatrix of A can be brought by a permutation of the rows and a permutation of the columns to the form

$$B = B_1 \dotplus \cdots \dotplus B_r, \tag{1}$$

where, for each i, B_i is a rectangular matrix of rank 1, with all its elements positive and such that the sum of squares of all elements of B_i is 1. Equation (1) means that B is obtained by laying out successively the rectangular blocks $B_1, \cdots, B_r$ with the lower right corner of B_i attached to the upper left corner of B_{i+1}, and with zeros filling in the entire matrix outside these blocks.

4873. *Proposed by W. R. Utz, University of Missouri*

Let S be a connected topological space in which there is a relation $<$ which satisfies: (1) S has a least and a greatest element relative to $<$, (2) $<$ satisfies the trichotomy law, (3) $<$ is continuous. In other words, if $x<y$ in S, then there exist neighborhoods N_x and N_y of x and y, respectively, such that $z \in N_x$ implies $z<w$ for each $w \in N_y$.

Show that any continuous transformation of S into itself has a fixed point and give an example of S which is not an arc.

4874. *Proposed by D. J. Newman, AVCO Research, Wilmington, Mass.*

Let c_n be a given sequence of constants such that $c_n = O(1/n)$, $n = 1, 2, \cdots$. Show that there is a bounded function $f(x)$ with c_n as its positive Fourier coefficients [*i.e.*, such that

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx = c_n\Big].$$

SOLUTIONS

Characterization of the Additive Group of Rationals

4822 [1959, 66]. *Proposed by J. deGroot, University of Amsterdam, Netherlands*

Prove that the additive group R of rational numbers is (up to isomorphisms) the only group satisfying the following conditions: (1) R is abelian, (2) R is infinite, and (3) every endomorphism (that is a homomorphic mapping of R into itself) is either an automorphism or a mapping on the null element.

Solution by D. G. Cantor, University of California, Los Angeles. Let G be a group satisfying (1), (2), and (3). By property (3) every nonzero endomorphism of G has an inverse and hence the endomorphisms of G form a division ring.

Let F be the prime field of this division ring. It is easy to verify that G is a vector space over F; and that it is one-dimensional. Otherwise there would be non-trivial endomorphisms of G onto lower dimensional subspaces which would not be automorphisms. Now F is isomorphic to either the integers, mod a prime, or the rationals. In the first case G would have only p elements contrary to (2); thus F is isomorphic to the field of rationals. Now every element of G is of the form $g_0 f$, for g_0 a fixed element of G and f an element of F. Let f' be the image of f in the field of rationals. Then $g_0 f \leftrightarrow f'$ is the desired isomorphism.

Also solved by H. F. Bechtell, Frederick Cunningham, Jr., C. C. Faith, Leopold Flatto, E. R. Gentile, Franklin Haimo, Melvin Henriksen, P. D. Hill, J. M. Horváth, L. E. Ward, Jr., J. V. Whittaker, R. J. Wisner and Charles Osgood, and the proposer.

Editorial Note. Several solvers referred to I. Kaplansky, *Infinite Abelian Groups*, in particular p. 10. Horváth finds the result in a theorem of T. Szele, *Die Abelschen Gruppen ohne eigentliche Endomorphismen*, Acta. Sci. Math., Szeged, vol. 13, pp. 54–56. See also J.-P. Serre, *Sur un théorème de T. Szele*, same volume, pp. 190–191.

By an analysis somewhat similar to the above solution, Gentile proves the more general result: *Let R be a commutative integral domain with identity and let G be an R-module such that every R-endomorphism of G is either an automorphism or a null map; then either G is 0 or G is R-isomorphic to the field of quotients of R/I for some prime ideal I of R.*

Derivative of Powers of a Matrix

4823 [1959, 66]. *Proposed by G. Matthews, St. Dunstan's College, Catford, England*

Let X be a lower-semi-matrix whose elements x_{ij} $(i, j = 1, 2, \cdots)$ are independent variables if $j \leqq i$ and zero if $j > i$, and let D be the differential operator matrix defined by $D_{ij} = \partial/\partial x_{ij}$ if $j < i$, $D_{ij} = 0$ if $j > i$ and $D_{ii} = \sum_{k=1}^{i} \partial/\partial x_{kk}$. Prove that $D(X^2) = 2X$, and hence by induction that $D(X^n) = nX^{n-1}$, where n is any positive integer greater than 2.

Solution by Leonard Carlitz, Duke University. We show first that $DX = I$. Indeed

$$DX = \Big(\sum_{j \leq k \leq i} D_{ik} x_{kj} \Big),$$

$$\sum_{j \leq k \leq i} D_{ik} x_{kj} = \sum_{j \leq k < i} \frac{\partial}{\partial x_{ik}} x_{kj} + \sum_{k=1}^{i} \frac{\partial}{\partial x_{kk}} x_{ij} = 0 + \delta_{ij}.$$

Secondly, since D_{ij} is linear in $\partial/\partial x_{rs}$, it is easily shown that

$$DAB = A \cdot DB + DA \cdot B,$$

where A and B are arbitrary square matrices whose elements are functions of x_{rs}. Therefore, by induction,

$$DX^n = X^{n-1} \cdot DX + DX^{n-1} \cdot X = X^{n-1} I + (n-1) X^{n-2} X = nX^{n-1}$$

for all integral $n \geqq 1$.

Also solved by R. D. Gordon, José M. Isidro, D. C. B. Marsh, B. E. Rhoades, D. H. Trahan, and the proposer.

Conformal Mapping

4824 [1959, 66]. *Proposed by D. J. Newman, AVCO Research and Development, Wilmington, Mass.*

Let $ABCD$ be a rectangle, $AB=1$, $BC=2$. Suppose that it is conformally mapped onto the upper half plane. If $A\to a$, $B\to b$, $C\to c$, $D\to d$, exhibit an elementary relation between a, b, c, d.

Solution by the proposer. We begin with an auxiliary mapping: Let $X=$ midpoint of BC and $Y=$ midpoint of AD. Map triangle ABX onto the semi-circle $|z-1| \leqq 1$, Im $z \geqq 0$, with $X\to 0$, $B\to 1$, $A\to 2$. By the reflection principle, if we invert through this semi-circle, this mapping can be continued to a mapping of $ABXY$ to the upper half plane with $Y\to\infty$.

Now apply the square root to this mapping and we get a mapping of $ABXY$ onto the first quadrant and so, again by the reflection principle, this can be continued to a mapping of $ABCD$ onto the upper half plane. We find, in this case, $A\to\sqrt{2}$, $B\to 1$, $C\to -1$, $D\to -\sqrt{2}$.

Since the given mapping must be a bilinear function of this one and since this leaves the cross ratio invariant, we have

$$\{abcd\} = \frac{(b-a)(d-c)}{(d-a)(b-c)} = -\frac{3\sqrt{2}-4}{8}.$$

Also solved by E. John Burr making use of the properties of the Schwarz-Christoffel transformation.

A Result of Ramanujan

4826 [1959, 66]. *Proposed by M. S. Klamkin and L. A. Shepp, AVCO Research and Development, Wilmington, Mass.*

If $\phi(x)=x/1^2-x^3/3^2+x^5/5^2-x^7/7^2+\cdots$, express $\phi(1)$ in terms of $\phi(2-\sqrt{3})$, thus obtaining a more rapidly converging expansion.

Note by Emil Grosswald, University of Pennsylvania. In Ramanujan's paper, *On the integral* $\int_0^x t^{-1}\tan^{-1} t\,dt$, he obtains the result

$$2\phi(1) = 3\phi(2-\sqrt{3}) + \tfrac{1}{4}\pi\log(2+\sqrt{3}).$$

(See *J. Indian Math. Soc.*, VII, 1915, pp. 93–96; also *Collected Papers*, pp. 40–43.*) Ramanujan's treatment, with some details filled in, follows.

One observes that

$$\phi(x) = \int_0^x \frac{\arctan t}{t}\,dt. \tag{1}$$

* See also L. Lewin, *Dilogarithms and Associated Functions*, London 1958, p. 39.

Next one shows that, for $0<x<\pi/2$,

$$(2)\qquad \frac{\sin 2x}{1^2}+\frac{\sin 6x}{3^2}+\frac{\sin 10x}{5^2}+\cdots=\phi(\tan x)-x\log(\tan x)$$

holds. Since both sides of (2) approach zero as $x\to 0$, it is sufficient to show that their derivatives are equal, *i.e.*, using (1), that

$$(3)\qquad \frac{2\cos 2x}{1}+\frac{2\cos 6x}{3}+\frac{2\cos 10x}{5}+\cdots=-\log(\tan x),\ 0<x<\pi/2.$$

Since (3) is a known cosine Fourier expansion, (2) is proved. With $x=\pi/12$, $\tan x=2-\sqrt{3}$, (2) becomes:

$$\frac{1}{2}\left\{\frac{1}{1^2}+\frac{2}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}-\frac{2}{9^2}-\frac{1}{11^2}+\frac{1}{13^2}+\frac{2}{15^2}+\cdots\right\}$$

$$=\phi(2-\sqrt{3})-\frac{\pi}{12}\log(2-\sqrt{3}).$$

The left member is $2\phi(1)/3$ as is shown by rewriting it as

$$\frac{1}{2}\left\{\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\cdots\right\}+\frac{3}{2}\left\{\frac{1}{3^2}-\frac{1}{9^2}+\frac{1}{15^2}+\cdots\right\}.$$

The desired result now follows immediately.

Also solved by L. Carlitz, Y. L. Luke, K. K. Norton, C. C. Yalavigi, and the proposers.

Inverse Functions

4827 [1959, 67]. *Proposed by J. Gallego-Diaz, Vanderbilt University*

If the development of the function $y=f(x)$ is given by

$$y=a_1x+a_3x^3+a_5x^5+\cdots,$$

find a function y knowing that if we invert the series we get

$$x=a_1y-a_3y^3+a_5y^5-\cdots.$$

I. *Solution by A. J. Kokar, School of Mines and Industries, Adelaide, Australia.* Noting that $a_1^2=1$ and that it is enough to consider $a_1=1$, we can put the given equations in the form

$$(1)\qquad \begin{aligned} &y=x(1+A_2x^2+A_4x^4\cdots), && x=y(1-A_2y^2+A_4y^4-\cdots),\\ &y/x=t=G(x^2), && x/y=1/t=G(-y^2); \end{aligned}$$

or we can rewrite them using the inverse function as

$$x^2=F(t),\qquad -y^2=F(1/t),$$

whence, upon dividing, we have

$$F(1/t) = -t^2F(t). \tag{2}$$

To solve (2), we may set $F(t)=k^2D(t)E(t)$, k^2 constant, $D(1/t)=t^2D(t)$, $E(1/t)=-E(t)$, and seek satisfactory functions D and E. Simple examples are

$$D(t) = 1/t \quad \text{and} \quad E(t) = (t - 1/t)^{2n+1},$$

$$x^2 = F(t) = k^2(t - 1/t)^{2n+1}/t.$$

Replacing t by y/x we have

$$(xy)^{2n+2} = k^2(y^2 - x^2)^{2n+1}. \tag{3}$$

For $n=0$ this gives

$$y = x(1 - x^2/k^2)^{-1/2} = x\sum_{j=0}^{\infty}\binom{2j}{2}\left(\frac{x}{2k}\right)^{2j},$$

$$x = y(1 + y^2/k^2)^{-1/2} = y\sum_{j=0}^{\infty}(-1)^j\binom{2j}{j}\left(\frac{y}{2k}\right)^{2j}.$$

which fulfills the requirements of the problem.

It is evident that there are infinitely many solutions with different values of n and with different choices of D and E. Furthermore, the function F can be factored in other ways, e.g.,

$$F(t) = kA(t)B(t)C(t),$$

$$A(1/t) = tA(t), \qquad B(1/t) = -tB(t), \qquad C(1/t) = C(t).$$

An entirely general form for the solution seems improbable.

II. *Solution by Leonard Carlitz, Duke University.* A solution of the problem is furnished by the gudermannian $y=\mathrm{gd}x$ (see, for example, Wilson, *Advanced Calculus*, p. 6) defined by $\sinh x=\tan y$ which has the property $\mathrm{gd}ix=i\mathrm{gd}^{-1}x$. We have at once

$$x = \mathrm{gd}^{-1}y = \log(\sec y + \tan y), \qquad dx/dy = \sec y.$$

From the known expansion of $\sec y$ there results

$$x = \sum_{n=0}^{\infty}(-1)^n\frac{E_{2n}}{2n+1}\frac{y^{2n+1}}{(2n)!},$$

$$y = \sum_{n=0}^{\infty}\frac{E_{2n}}{2n+1}\frac{x^{2n+1}}{(2n)!},$$

where E_{2n} are the Euler numbers in the even suffix notation.

Also solved by R. D. Gordon, and the proposer.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

The New Mathematics. By Irving Adler. John Day, New York, 1958. 187 pp. $3.75.

This is a delightful little book. The forward declares that the book is addressed to the average reader. It is the opinion of the reviewer that the average reader would enjoy much of the book but would have some difficulty in understanding everything in the book. Although explanations are as elementary as possible, the average reader would probably find difficulty following the discussions of neighborhoods and vector spaces.

For the teacher of secondary school mathematics and the capable high school student the book should prove very valuable. For the teacher who has studied some modern algebra the book will give a glimpse of some "old friends." The teacher lacking this background will certainly gain a great deal by reading the book. He will have a better understanding of the complex number system and will find some meaning in such terms as "mapping," "sets," and "structure."

The reader is introduced to a variety of mathematical structures: groups, rings, fields, vector spaces, and topological spaces. Most of the concepts are introduced first by the use of familiar examples in the number systems and then by less familiar examples such as residue classes and matrices.

Few proofs are included, so the author moves along rapidly and the book is read easily. Some exercise lists are included which would assist the reader in understanding new ideas.

The book should create a desire on the part of many readers for further study. A bibliography is included.

The author does a successful job of creating a feeling of expectancy in the early chapters. He convinces the reader that he is seeking something and that he does not find it until the last few pages—a number system which would solve all algebraic equations.

In the closing paragraph the author notes that although the names of certain mathematical structures are new, they are closely related to such familiar things as addition and multiplication. He concludes, "Although the world of modern mathematics is a new world in many ways, it has never lost contact with the old world of number and space from which it has grown."

Teachers of secondary school mathematics and laymen with a special interest in mathematics will enjoy this book. It should find a place in high school libraries.

MARIE S. WILCOX
Thomas Carr Howe High School
Indianapolis, Indiana

Vorgriechische Mathematik I. Vorgeschichte und Ägypten. By Kurt Vogel. Mathematische Studienhefte, Hefte 1. Hermann Athen and Georg Wolff, editors. Hermann Schroedel Verlag KG, Hannover, Germany, 1958. 80 pp. 8 D.M. (About $2).

It was the late George Sarton, historian of science, who once asked, "How could we really appreciate the traditions which give our lives their meaning, if we did not continue them. . . . The past is full of beauty, but less so however than the future." Dr. Vogel, evidently, has this for his thesis in his latest work written primarily for gymnasium students in an effort to aid their understanding of both ancient culture and the influence of ancient mathematics upon that of the present day.

The first part of this excellently written and illustrated paper-back volume deals with prehistoric geometry and numbers in Europe, Crete, Mohenjo-Daro, and Sumer. The mathematics of the Egyptians, in which Vogel has been working for many years, is brought up to date with the results of recent Egyptological research. The Egyptian number system, metrology, fundamental operations, arithmetic and algebraic problems, and geometry are explained adequately for the layman without loss of mathematical detail.

Each of the two parts of the book has a set of exercises and literature references; a full index is at the end.

Vogel's interesting work should serve eminently the purpose for which it was intended. With a better understanding of its history, mathematics will emerge as one of the most important cultural interests of our time.

MARTIN LEVEY
Temple University

Introduction to Functional Analysis. By A. E. Taylor. Wiley, New York, 1958. xvi+423 pp. $12.50.

This book attempts to fill a large gap in the textbook literature. Although the book (as is stated in the title) is an introduction, it is not exactly elementary, being at a much higher level than, for example, Kolmogorov and Fomin (Albany, N. Y., 1957). The book is thorough. A degree of completeness is achieved such as is seldom realized in a textbook. There is a vast number of exercises, many of which are important theorems.

After preliminary chapters on vector spaces and the elements of general topology, there is a longer chapter entitled "topological linear spaces." The scope of the discussion is essentially confined to normed spaces so that the title is somewhat misleading. The general theory is continued in chapter 4, which contains basic theorems on metrizable linear spaces such as the operator inverse theorem, the closed graph theorem, and the principle of uniform boundedness. Also included are representations for the duals of special Banach spaces. The next two chapters, which seem to be the core of the book, are concerned with spectral theory, first for Banach spaces in general, and then for bounded oper-

ators in Hilbert space. The last chapter on "integration and linear functionals" gives one the impression of being an afterthought. Here, T is a locally compact Hausdorff space, $C_\infty(T)$ the space of continuous functions, with compact support, on T and $\|x\| = \sup_{t \in T} |x(t)|$. ($C_\infty(T)$ is not complete, in general.) It is proved that the dual, with the strong topology, is isomorphic, as a Banach space, to the Banach space of finite, regular, signed Borel measures on the Borel sets in T. The isomorphism is given by $x'(x) = \int x d\mu$.

The book is to be commended as a worthy intellectual accomplishment. Time will tell its value as a text for graduate courses. It certainly merits a trial.

CASPER GOFFMAN
Purdue University

Topological Analysis. By Gordon Thomas Whyburn. Princeton University Press, Princeton, N. J., 1958. xii+119 pp. $4.00.

Topological analysis consists of those basic theorems of analysis, especially of the functions of a complex variable, which are essentially topological in character, developed and proved entirely by topological and pseudotopological methods. This book expands and gives the details of the program outlined by the author in his lecture "Introductory Topological Analysis" [See *Lectures on Functions of a Complex Variable*, University of Michigan Press, 1955, pp. 1–14.]

Professor Whyburn has been deeply interested in interior or open mappings from the time of their introduction by Stoïlow [See *Principes Topologiques de la Theorie des Fonctions analytiques*, Paris, 1938.] and, in particular, in their intimate connection with analytic functions. Stoïlow early recognized lightness and openness as the two fundamental topological properties of the class of all nonconstant analytic functions. Whyburn, in the intervening years, has made deep and penetrating discoveries in the field of light interior mappings. In particular, he has characterized their action on 2-dimensional manifolds.

The present text develops the surprisingly small amount of background necessary for an understanding of this material, and then studies the circulation index of a mapping about a point. Using this concept the lightness and openness properties of mappings generated by differentiable functions of a complex variable are established. Open mappings are studied in a general topological setting leading up to the complete local analysis of a light open mapping on a 2-dimensional manifold. Global relationships of the Hurwitz Formula type are obtained for the compact manifold case. Rouche's Theorem and other results of this type are proved. One of the outstanding results of the book is the proof of the Hurwitz Theorem on sequences, in its full generality, without using the differentiability of the limit function. The proof, as do most others in the text, uses almost none of the results or techniques of classical analysis. Poles of a function are defined without series expansion.

Analysts who have the patience to study the topological background material (pages 1–40, including a complete point-set proof of the Jordan Curve Theorem), or who have read the same material in Whyburn's book on *Analytic*

Topology [New York, 1942, American Mathematical Society, Colloquium Publications, vol. 28], will find their efforts richly rewarded in the remainder of the book. Professor Whyburn has done mathematics a great service by providing this textbook as a beacon light to guide forward in their research both analysts and topologists who are interested in obtaining further results in both fields.

DICK WICK HALL
Harpur College

An Introduction to Combinatorial Analysis. By John Riordan. Wiley, New York, 1958. x+244 pp. $8.50.

This elegant treatise on the art and science of counting will undoubtedly become a standard text and reference. Chapter 1 treats the usual theory of permutations and combinations, with emphasis on the use of generating functions. The theory of those is developed more fully in Chapter 2, which includes a discussion of the important and oft-recurring Bell polynomials. Chapter 3, on the principle of inclusion and exclusion, is followed by an excellent treatment of permutations in cyclic representation. After a survey of distributions and occupancy (Chapter 5), the author takes up, in Chapter 6, partitions, compositions, and the enumeration of trees and graphs. Here he presents the beautiful and powerful theorem of Pólya. The last two chapters are an extended development of permutations with restricted position, in which field the author has made many contributions.

Some outstanding characteristics of this book are its consistent and systematic use of the generating function as a powerful tool; its wealth of applications, both in the text and in the extensive problem sections; its many useful tables; its clarity of presentation; its accuracy and excellent format. There were only a few instances (*e.g.*, di Bruno's formula) where the development could have been more direct and illuminating. This reviewer would have liked to see at least a statement of Ramanujan's beautiful congruence theorems and of the asymptotic formula for $p(n)$. Indeed, there is a conspicuous lack, throughout, of order of magnitude considerations. But these are only minor flaws in an important work covering an important field.

N. J. FINE
University of Pennsylvania and
Institute for Adanced Study

Difference Methods for Initial-Value Problems. By Robert D. Richtmyer. Interscience, New York, 1958. xii+238 pp. $7.25.

When formulated mathematically, many of the most important problems in theoretical physics lead to a partial differential or integro-differential equation whose solution, subject to appropriate subsidiary conditions, specifies the state of the physical system under study. Although some of the simpler problems (such as the vibrating string and heat flow equations of the elementary textbooks) have explicit solutions in terms of Fourier series and integrals, most

realistically formulated problems allow only approximate numerical solutions. Under such circumstances, the continuous space and time variables are made discrete and systems of finite-difference equations are solved in a step-wise fashion to furnish approximations to the solution of the corresponding continuous problem.

Studying the book under review is an excellent way to learn of the present state of knowledge concerning such finite-difference methods. Since the beginnings of the theory in the classic paper by Courant, Friedrichs, and Lewy [*Über die partiellen Differenzengleichungen der mathematischen Physik*, Math. Annalen, vol. 100, 1928, pp. 32–74] much progress has been made, especially at the wartime atomic energy laboratories and at New York University's Institute of Mathematical Sciences. Nevertheless, high-speed computers have been and are still being used to solve highly complex problems in the absence of rigorous proof that the machine methods do what they are supposed to do. Convergence proofs and error estimates are not available for many important problems. These get solved numerically by methods based more on empirical tests and intuition than on mathematical demonstration.

Theory and practice are both treated in this book. Convergence of the difference equation solution to the true solution of the continuous problem, rates of convergence and stability of various difference equation systems, and methods of solving implicit difference systems are discussed in Part I. Here the main theorem is the equivalence between stability and convergence proved by Peter Lax. The general theory, expressed in the language of linear operators, is then specialized to pure initial-value problems with constant coefficients, and the von Neumann condition for stability is derived and studied in great detail. Part II (approximately two-thirds of the book) is devoted to a discussion of difference-equation methods that have proved useful for initial-value problems arising in heat flow, neutron diffusion, radiation transfer, sound waves, elastic vibrations, and fluid dynamics.

SAMUEL GOLDBERG
Oberlin College

Mathematics in Fun and in Earnest. By Nathan Altshiller Court. Dial Press, New York, 1958. 250 pp. $4.75.

Altshiller Court's book is an illustration of his own credo: mathematics in earnest should be fun; mathematics in fun may be earnest. The book has no index, but a five-page table of contents furnishes reasonable orientation. The enthusiasm of the author carries the reader along. Everything is grist to Court's mill: philosophy of mathematics, logic and set-theory, paradoxes, basic concepts, games and puzzles and other topics. The result is a buffet where you sample one dish after another for aroma and flavour. References at the end of each chapter offer opportunity for further investigation. In many cases this

seems indispensable, most markedly perhaps in connection with the following passage (p. 65):

"Hilbert labored for many years trying to produce a proof that the axioms of his 'Grundlagen der Geometrie' satisfied this requirement [of being logically consistent with each other]. But all his persistent zeal and his enormous intellectual resources proved unequal to the task, though he could find some personal consolation in the proposition, proved by K. Goedel in 1931, that the Grundlagen could not yield a proof of its own consistency. Georges Bouligand (*Le déclin des absolus mathématico-logiques*, Paris, 1949, p. 16) formulated the argument as follows: 'To find within a body of doctrine G a proof that G is consistent is impossible, for to accept the validity of such a proof is to concede to a part of G a special privilege: an abusive procedure, if the coherence of G as a whole is in doubt.' *Simple and obvious, David Hilbert to the contrary notwithstanding.*" (Italics mine).

Since our author's main scientific interest has been consistently in the domain of geometry, as is well known to readers of this MONTHLY, it is not surprising that his condensed discussions of this field, for example on Euclidean and on projective geometry, and on the role of the infinite in these disciplines, should be rewarding in their clarity.

Among mathematical puzzles the graphical solution of the classical "pouring problems" is worthy of notice (credited to Perelman). One runs across intriguing statements such as "[the mathematical] proofs are not an integral part of the mathematical doctrine." In "Mathematical Asides" we find a quantitative discussion of the fact that men and beasts lost or blindfolded tend to move in circles, based on the assumption of slight differences in the lengths of legs. "The Perplexities of a Potato Pusher," pp. 171–181, may seem a little out of place, both in form and content, and two problems in elementary geometry may not have gained by the forced humorous treatment.

On p. 239 we read that Vinogradov proved in the thirties of our own century that any odd number (instead of: every sufficiently large odd number) is the sum of three primes.

Very careless proofreading is hard to excuse. Of many misprints and inaccuracies, I mention only misspelled author names: Halstead for Halsted (consistently); Lengendre for Legendre (p. 161); Frenel for Fresnel (p. 178); Danzig for Dantzig (p. 134); Appollonius for Apoll. (p. 102); Non-Arguesian for Non-Desargu. (p. 179).

All in all, we must be grateful to our author for expressing without inhibition his thoughts on a wide range of mathematical and philosophical topics. Even in disagreeing with him, one appreciates his adventurous spirit. The words of Goethe come to one's mind: Wer Vieles bringt wird Manchem etwas bringen.

AUBREY J. KEMPNER (Emeritus)
University of Colorado

Fundamentals of Mathematics, Rev. Ed. By M. Richardson. Macmillan, New York, 1958. xviii+507 pp. $6.50.

This book, as did the original edition (for a review of the original see the Aug.-Sept. 1941 MONTHLY), "attempts to provide a suitable terminal course for students of the arts and social sciences, stressing the fundamental concepts and applications of mathematics rather than its formal techniques." In this revision some sections have been rewritten (especially the section on probability), and sections have been added on: electronic computers, information theory, the algebra of propositions and truth tables, the application of Boolean algebra to electrical networks, political structures, inequalities, linear programming, the theory of games of strategy, and individual and social preferences.

If one looks over these additions and the chapter titles it may be observed that the topics come quite close to those in the experimental texts for the freshman Universal Mathematics course recommended by the Committee on the Undergraduate Program of the Association. Thus, those teachers looking for a useable text which covers approximately the material suggested by the C.U.P. might well consider this one.

Since Richardson's 500-page book also "covers" topics such as topology, non-Euclidean geometry, and transfinite cardinals, which are not suggested by the C.U.P., it is fairly clear that much of the coverage must lack depth. The lack of depth may be gauged from the following items. Many properties of the natural numbers are deduced in the first part of the book but induction is not mentioned until a chapter near the end. (This actually may be wise pedagogically.) Complex numbers are introduced but the student is not told what it means to add or multiply two of them. A rational fraction is defined to be a symbol. (A footnote explains that this is not above logical reproach.) The chapter on trigonometry does not discuss addition formulas or identities. "Random sample" is not defined (in fact, it is confused with "representative sample") and much of the chapter on statistics seems to be much "thinner" than it should be for a terminal math course for social scientists. There are too few problems in the statistics chapter.

Several of the newly added sections are excellent introductions to some of the new applications of mathematics and the book should succeed in giving students a good "feeling" for the nature of mathematics and some of its applications. Whether a teacher will find it suitable as a text or not will depend on whether he considers the attainment of this feeling adequate compensation for loss of a deeper treatment of (necessarily fewer) topics.

All in all, the book is a valuable attempt at a compromise between talking about mathematics and doing mathematics.

FRANK L. WOLF
Carleton College

An Introduction to Mathematics. By Alfred North Whitehead. Oxford University Press, New York, 1958. A Galaxy Book. v+191 pp. $1.50.

This small volume, which appeared in England in 1911, was not published in this country until 1948, a year after its author's death. At that time new drawings were prepared, but the text remained unchanged, and here appears in "paper-back" form.

Written "not to teach mathematics, but to enable students from the beginning of their course to know what science is about" by freeing it from a mass of technical details, the book deals with the basic ideas of those topics of classical mathematics that were of interest to the budding scholar of 1911: the nature of mathematics, variables, symbolism, conic sections, geometry, trigonometry and others. In fact the budding scholar of today, struggling with the notions of continuity and limits, might be directed to the sections on functions, the differential calculus, and infinite series.

But it is its treatment of the development of scientific ideas through the centuries, showing why they languished or came to fruition when they did, as well as its lucidity of thought and expression, that gives the book its chief claim to distinction. And for the reader who does not know Whitehead, his warmth, his simplicity, his quiet humor, his utter lack of pretension, it will provide a rewarding introduction to a man as well as to a philosopher and scientist.

BESS E. ALLEN
Wayne State University

Lectures on Ordinary Differential Equations. By Witold Hurewicz. Technology Press (M.I.T.) and Wiley, New York, 1958. xvii+122 pp. $5.00.

This monograph constitutes a rigorous and lively introduction to the theory of ordinary differential equations. The approach is essentially geometric and extremely appealing to the intuition. The book was prepared from mimeographed notes left by Professor Hurewicz under the editorship of Professor Levinson. The reviewer feels that the editor did an excellent job of completing these notes into book form without losing any of Professor Hurewicz's style. Chapter I studies the first order system and starts with direction fields, goes into approximate solutions, existence and uniqueness theorems and extensions of these theorems. Chapter II is concerned with systems of first order equations, Lipschitz conditions, properties of solutions, and the reduction of higher ordered systems. In Chapter III matrix notation is used and the author makes a very careful presentation of fundamental systems and their properties, the Wronskian, Green's function, eigenvalues and eigenfunctions. Chapter IV introduces autonomous systems and, for the linear systems, the singularities are classified in terms of the local behavior of the characteristics. Certain nonlinear systems are reduced to this situation under carefully stated conditions. The last chapter (V) deals with solutions of autonomous systems in the large. The principal result

here is the Poincaré-Bendixson Theorem. Poincaré's index is used to characterize orbital stability of limit cycles and simple singularities. In the opinion of the reviewer, the careful and lucid way in which this book is written puts it well within the grasp of the senior mathematics student.

PASQUALE PORCELLI
Mathematics Research Center (U. S. Army)
The University of Wisconsin

BRIEF MENTION

An Introduction to the Geometry of N Dimensions. By D. M. Y. Sommerville. Dover, New York, 1958. xvii+196 pp. $1.50.

This reprint first appeared in 1929.

The Supervisor of Mathematics, His Role in the Improvement of Mathematics Instruction. National Council of Teachers of Mathematics, Washington 6, D. C., January, 1959. 10 pp. 15 cents each; 10 or more copies, 10 cents each.

A very timely publication, since more and more school systems are now providing the services of a separate supervisor of mathematics. Some of these posts are currently being filled by persons with little or no graduate training *in mathematics;* a deplorable situation

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NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor E. B. Mode, Wellesley, Massachusetts, represented the Association at the inauguration of President A. S. Knowles of Northeastern University on September 8, 1959.

University of Colorado: Professor B. W. Jones will be on leave at the University of Puerto Rico until June 1960; Professor L. J. Mordell, Cambridge University, England, will be a Visiting Professor until June 1960.

University of Georgia: Professor M. K. Fort, Jr., has been appointed Head of the Mathematics Department; Professor A. C. Cohen, Jr. has been appointed Director of the newly established Institute of Statistics; Assistant Professor R. P. Hunter will be on leave for one year to Oxford, England; Assistant Professor B. J. Ball, University of Virginia, has been appointed Associate Professor; Assistant Professor J. G. Horne, Jr., University of Kentucky, has been appointed Assistant Professor.

Massachusetts Institute of Technology: Professors Warren Ambrose, C. C. Lin, I. M. Singer, and Assistant Professor M. L. Minsky will be on leave during 1959–60; Associate Professor G. B. Whitham, New York University, has been appointed Professor; Dr. Sigurdur Helgason, University of Chicago, has been appointed Assistant Professor and will be on leave at Columbia University during the academic year 1959–60; Assistant Professor D. M. Kan, Hebrew University, Jerusalem, Israel, and Dr. G. C. Rota, Har-

vard University, have been appointed Assistant Professors; Dr. J. G. Clunie, Imperial College, University of London, will be a Visiting Assistant Professor; Dr. R. J. Gribben, Sloan Foreign Postdoctoral Fellow during 1958–59 will be Lecturer; Drs. Peter Dombrowski, Bonn University, Germany, G. H. Fullerton, Queen's University, Belfast, Ireland, Harry Furstenberg, Princeton University, Walter Koppelman and P. C. Shields, Yale University, W. G. Strang and E. O. Thorp, University of California, Los Angeles, have been appointed C. L. E. Moore Instructors; Drs. G. F. Feeman, Muhlenberg College, Patrick Gallagher, Princeton University, and J. D. Thomas, Los Alamos Scientific Laboratories, have been appointed Instructors; Drs. R. F. Chisnell, University of Manchester, England, Genjiro Fujisaki, University of Tokyo, Japan, Yosutaka Sibuya, University of California, Los Angeles, and Ankira Sakurai, New York University, have been appointed Research Associates; Dr. Takayoshi Mitsui, University of Tokyo, Japan, has been awarded a Sloan Foreign Postdoctoral Fellowship in the School for Advanced Study at M.I.T.

University of Wisconsin: Professor P. C. Hammer, Director of the Numerical Analysis Laboratory, has been given a research grant by the University Research Committee for the year 1959–60, and will spend the year in Europe. In his absence Professor E. A. Robinson will be Acting Director.

Mr. A. C. Ahlin, Massachusetts Institute of Technology, has accepted a position as Mathematician with the Boeing Airplane Company, Renton, Washington.

Mr. R. C. Anderson, Sr., Alabama College, has accepted a position as Mathematician with the Army Ballisic Missile Agency, Redstone Arsenal, Huntsville, Alabama.

Dr. K. I. Appel, University of Michigan, has accepted a position with the Institute for Defense Analyses, Princeton, New Jersey.

Assistant Professor R. R. Archer, Massachusetts Institute of Technology, has been appointed Assistant Professor at the University of Massachusetts.

Assistant Professor W. G. Bade, University of California, Berkeley, has been promoted to Associate Professor.

Mr. C. R. Berndtson, AVCO Manufacturing Corporation, Stratford, Connecticut, has been appointed Mathematics Teacher at Maine Central Institute, Pittsfield, Maine.

Mr. J. A. Berton, University of Illinois, has been appointed Instructor at Indiana State Teachers College.

Associate Professor A. A. Blank, University of Tennessee, has been appointed Associate Professor at the Institute of Mathematical Sciences, New York University.

Professor F. E. Bowling, Lincoln Memorial University, has been appointed Professor and Head of the Department of Mathematics at Tennessee Wesleyan College.

Assistant Professor J. R. Boyd, Lamar State College, has been appointed Assistant Professor at Arlington State College.

Dr. D. G. Brennan, Massachusetts Institute of Technology, has been appointed Assistant Group Leader at M.I.T., Lexington Laboratory.

Mr. J. T. Brogan, Mississippi State University, has accepted a position as Research Engineer with Autonetics, Downey, California.

Brother L. R. Foy was appointed President of Marian College, Poughkeepsie, New York.

Mr. Thomas Courtney, Texas Technological College, has accepted a position as Junior Engineer with the Philco Corporation, Philadelphia, Pennsylvania.

Dr. C. H. Cunkle, Cornell Aeronautical Laboratories, Inc., Buffalo, New York, has been appointed Associate Professor at Utah State University.

Assistant Professor Frederic Cunningham, Jr., Wesleyan University, has been appointed Associate Professor at Bryn Mawr College.

Dr. G. L. Davey, Edwards Air Force Base, California, has been appointed a Member of the technical staff at Hughes Aircraft Company, Culver City, California.

Mr. E. R. Deal, University of Michigan, has been appointed Assistant Professor at Colorado State University.

Mr. Walter Ehrenpreis, University of Connecticut, has been appointed Assistant Professor at Trenton State College.

Dr. H. T. Engstrom, National Security Agency, Washington, D. C., has been appointed Vice President of Remington Rand, New York.

Professor Frances E. Falvey, Millikin University, has been appointed Dean of Milwaukee-Downer College.

Mr. C. K. Fendall, Boeing Airplane Company, Seattle, Washington, has accepted a position as Staff Member, Mathematical Services Department, Aeronutronic, Newport Beach, California.

Miss Barbara A. Flores, Baylor University, has accepted a position as Chemical Technician with the Dow Chemical Company, Freeport, Texas.

Mr. George Forester, Ross Laboratories, Columbus, Ohio, has been appointed Science Supervisor for the Catlin Gabel School, Portland, Oregon.

Assistant Professor D. J. R. Foulis, Lehigh University, has been appointed Assistant Professor at Wayne State University.

Dr. Stanley Frank, University of Florida, has been appointed Teacher and Acting Head of the Mathematics Department at St. Johns River Junior College, Palatka, Florida.

Mr. R. B. Gardner has been appointed Lecturer at Columbia University.

Dr. J. M. Gary, California Institute of Technology, has been appointed Assistant Professor at Harvey Mudd College.

Dr. R. E. Gaskell, Boeing Airplane Company, Seattle, Washington, has been appointed Professor at Oregon State College.

Associate Professor Samuel Goldberg, on leave from Oberlin College, has been appointed Visiting Associate Professor to the Graduate School of Business Administration, Harvard University.

Associate Professor F. M. C. Goodspeed, University of British Columbia, had been appointed Associate Professor at Laval University, Quebec.

Associate Professor Bernard Greenspan has been appointed Professor on his return to Drew University.

Dr. D. S. Greenstein, University of Michigan, has been promoted to Assistant Professor.

Professor Marshall Hall, Jr., Ohio State University, has been appointed Professor at the California Institute of Technology.

Dr. Carl Hammer, Sylvania Electrical Products, Inc., Needham, Massachusetts, has accepted a position as Administrator, Technical Projects Coordination, Radio Corporation of America, New York.

Professor J. R. Hanna, on leave from the University of Wichita, is Design Engineer II at the Martin Company, Denver, Colorado.

Mr. G. G. Harrington, Southern Methodist University, has accepted a position as Associate Research Engineer with the Boeing Airplane Company, Seattle, Washington.

Mr. W. G. Hazlett, Chance Vought Aircraft, Inc., Dallas, Texas, has accepted a position as Program Instructor for the System Development Corporation, Santa Monica, California.

Assistant Professor Henry Hiz, on leave from Pennsylvania State University, has been appointed Visiting Professor at the University of Pennsylvania.

Mr. R. H. Hoskins, John Hancock Mutual Life Insurance Company, Boston, Massachusetts, has been promoted to Associate Group Actuary.

Mr. A. R. Hyde, Kingswood School, West Hartford, Connecticut, has accepted a position as an Editor with the Ronald Press Company.

Mr. R. Q. Jennett, Convair, Fort Worth, Texas, has accepted a position as a Design Specialist with the Martin Company, Orlando, Florida.

Mr. B. A. Jensen, Dana College, has been appointed Assistant Professor at Nebraska Wesleyan University.

Mr. W. J. Jonsson, Defence Research Board, Operations Research Group, St. Hubert, Quebec, has been appointed a Lecturer at the University of Manitoba, Fort Garry, Manitoba.

Mr. N. M. Kendall, Kansas University, has accepted a position as a Mathematician with the Bendix Computer Division, Los Angeles, California.

Professor E. C. Kiefer, Millikin University, has retired and been given the title Professor Emeritus. He will be at Western Illinois University during 1958–59 as a Visiting Professor.

Mr. R. W. King, Vanderbilt University, has been appointed Assistant Professor at Mississippi College.

Mr. W. R. Knight, University of Toronto, has been appointed Assistant Professor at the University of New Brunswick, Fredericton.

Professor C. F. Kossack, Purdue University, has accepted the position of Manager of the newly formed statistics and operations research department of International Business Machines Corporation, Lamb Estate Research Center, Town of Cortlandt, New York.

Assistant Professor R. G. Kuller, Wayne State University, has been appointed Assistant Professor at Dartmouth College.

Mr. G. L. Lane, University of Kansas, has been appointed a Staff Member of the Sandia Corporation, Albuquerque, New Mexico.

Mr. B. L. Lercher, Pennsylvania State University, has been appointed an Instructor at the University of Rochester.

Mr. Shen Lin, Ohio State University, has been appointed Assistant Professor at Ohio University.

Associate Professor F. L. Lynch, Jr., Seton Hall University, has been appointed Chairman of the Mathematics and Physics Departments.

Mrs. Helen M. Marston, Educational Testing Service, Princeton, New Jersey, has been appointed Instructor at Douglass College of Rutgers, The State University.

Mr. C. L. McCarty, Jr., Georgia Institute of Technology, has been appointed Research Mathematician at the University of North Carolina Research Computation Center.

Assistant Professor J. H. McKay, Seattle University, has been appointed Associate Professor at Michigan State University Oakland, Rochester, Michigan.

Dr. R. D. McWilliams, Princeton University, has been appointed Assistant Professor at Florida State University.

Mr. H. L. Nelson, University of Kansas, has accepted a position as Mathematical Analyst with the Autonetics Division of North American Aviation, Los Angeles, California.

Mr. R. S. Nicholls, Miles Laboratories, Inc., Elkhart, Indiana, has been promoted to Director, Research Coordination.

Assistant Professor J. A. Nickel, Willamette University, has been appointed Associate Professor and Chairman of the Mathematics Department at Oklahoma City University.

Mr. G. A. Paxson, College of the Holy Names, has accepted a position as Mathematician with the Mathematical Services Group of the California Research Corporation, Richmond, California.

Professor W. T. Reid, Northwestern University, has been appointed Professor and Head of the Department of Mathematics at the State University of Iowa.

Dr. Azriel Rosenfeld, Ford Instrument Company, Long Island City, New York, has accepted a position as Staff Consultant with the Lewyt Manufacturing Corporation, Long Island City, New York.

Dr. C. V. L. Smith, Aberdeen Proving Ground, Maryland, has accepted a position as Chief, Data Systems Division, NASA Goddard Space Flight Center, Washington, D. C.

Mr. W. G. Spohn, Jr., Bowling Green State University, has been appointed Mathematician with the Applied Physics Laboratory, Johns Hopkins University, Silver Spring, Maryland.

Mr. D. C. Stevens, Ohio State University, has been appointed an Instructor at Alabama College.

Associate Professor W. L. Strother, University of Miami, has been appointed Professor and Chairman of the Mathematics Department, Miami University, Oxford, Ohio.

Professor G. L. Thompson, Ohio Wesleyan University, has been appointed Associate Professor in the Graduate School of Industrial Administration, Carnegie Institute of Technology.

Assistant Professor E. P. Tovani, University of Chicago, has accepted a position as Senior Research Engineer with Convair Astronautics, San Diego, California.

Mr. George Van Zwalenberg, University of California, Berkeley, has been appointed Instructor at Bowling Green State University.

Dr. Warren Weaver has retired from his position as Vice President for the Natural and Medical Sciences of The Rockefeller Foundation and has become Vice President of the Alfred P. Sloan Foundation.

Dr. Harold Weitzner, University of California, Berkeley, has been appointed Associate Research Scientist, Institute of Mathematical Sciences, New York University.

Dr. W. F. Whitmore, Special Projects Office, United States Navy, has accepted a position as Consulting Scientist on the staff of the Chief Scientist, Lockheed Missiles and Space Division, Sunnyvale, California.

Professor Hazel S. Wilson, Doane College, has been appointed Professor at Jacksonville University.

Mr. C. R. Woodrow, Oklahoma State University, has been appointed an Instructor at Austin College.

Professor F. L. Wren, George Peabody College, has been appointed Professor at San Fernando Valley State College.

Mr. C. E. Yingst, Lockheed Aircraft Corporation, Burbank, California, has accepted a position as Computer Programmer at the Thompson-Ramo-Wooldridge Products Company, Los Angeles, California.

Professor Emeritus R. E. Gaines, University of Richmond, died on June 19, 1959. He was a charter member of the Association.

Assistant Professor Lena E. Reynolds, Chapman College, died July 13, 1959. She was a member of the Association for thirty-six years.

NATIONAL ACADEMY OF SCIENCES—NATIONAL RESEARCH COUNCIL DIVISION OF MATHEMATICS

FELLOWSHIP AND RESEARCH OPPORTUNITIES

The Division of Mathematics calls attention to the fact that several foundations and offices offer a number of fellowships as well as financial support for basic research in mathematics during the year 1960–61. A partial list, with comments, is given below; other sources of support are given in the bulletin, "A Selected List of Major Fellowship Opportunities and Publications for Educational Support," available from the Fellowship Office, National Academy of Sciences—National Research Council, 2101 Constitution Avenue, Washington 25, D. C.

1. *National Science Foundation.* The National Science Foundation sponsors various fellowship programs in the sciences, including mathematics. Awards, available only to citizens of the United States, are made solely on the basis of ability.

A. *Predoctoral Fellowships*—offered annually in three programs.

(1) *Graduate* fellowships are awarded at the First Year, Intermediate, and Terminal Year levels of study. Applications for 1960–61 will be available in October 1959 from the National Academy of Sciences—National Research Council, 2101 Constitution Avenue, N.W., Washington 25, D. C., until the closing date, January 1, 1960. Award date—March 15, 1960.

(2) *Cooperative Graduate* fellowships are tenable at approximately 150 participating institutions. Application materials are available at, and are submitted through, the selected institution. For information and list of participating institutions, write The Fellowships Section, Division of Scientific Personnel and Education, National Science Foundation, Washington 25, D. C. Closing date for receipt of applications—November 6, 1959. Award date—approximately April 1, 1960.

(3) *Summer Fellowships for Graduate Teaching Assistants* make it possible for graduate teaching assistants to continue full-time academic study and/or research at approximately 150 participating institutions. Application materials are available at, and are submitted through, the teaching assistants' own institutions. For information and list of participating institutions write to the National Science Foundation. Closing date—December 11, 1959. Award date—approximately April 1, 1960.

B. *Postdoctoral Fellowships*

(4) *Postdoctoral* fellowships for recent recipients of the doctoral degree who desire additional advanced training preparatory to specialized scientific work. Application material available from the National Academy of Sciences—National Research Council. Two award periods: (1) Closing date for first competition—September 1, 1959. Award date—October 15, 1959. (2) Second award period from October 1959 to December 22, 1959. Award date—March 15, 1960.

(5) *Senior Postdoctoral* fellowships are awarded to scientists who have demonstrated ability and special aptitude for productive scholarship in the sciences and who have held the doctoral degree for a minimum of 5 years at time of application. Application material may be obtained from the National Science Foundation after May 15, 1959 for submission prior to October 5, 1959. Award date—December 7, 1959.

C. *Faculty Fellowships*

(6) *Science Faculty* fellowships are awarded to college teachers of science (including mathematics) who plan to continue teaching and wish to increase their competence as teachers. Eligibility requirements include a baccalaureate degree and three (3) years of full time experience at the collegiate level. Application material may be obtained from the National Science Foundation after May 15, 1959 for submission prior to October 5, 1959. Award date—December 7, 1959.

(7) *Summer Fellowships for Secondary School Teachers of Science and Mathematics* provide opportunities for teachers of high ability to pursue individually planned programs at the graduate level. Tenures from one to three summers are available. Information and application material will be available in October 1959 from Secondary School Fellowships, American Association for the Advancement of Science, 1515 Massachusetts Ave., N.W., Washington 5, D. C. Closing date early January 1960. Award date—March 15, 1960.

D. *Research Grants.* The National Science Foundation also supports basic research in the mathematical sciences by means of grants. Proposals for such support are ac-

cepted at any time. They should be submitted about six months before the applicant wishes to receive notification of the Foundation's decision. Instructions for the preparation of proposals, contained in a booklet entitled, "Grants for Scientific Research," may be obtained upon request from the Program Director for Mathematical Sciences, National Science Foudation.

2. *Office of Naval Research.* The Office of Naval Research, through contracts with universities and other organizations, supports basic research in broadly selected fields of mathematics. Proposals should be directed to the Mathematics Branch, Office of Naval Research, Washington 25, D. C. In addition, postdoctoral research associateships in pure mathematics are established under contracts with the ONR at selected universities. For details and application forms write to the above address.

3. *Air Force Office of Scientific Research.* The Air Force Office of Scientific Research supports research in mathematics directly through contracts with colleges, universities, foundations and industrial laboratories. Interested research mathematicians are encouraged to submit proposals, through their organizations, for research in mathematical fields in which they specialize. Proposals should be mailed to the Commander, Air Force Office of Scientific Research, Attn.: Director of Mathematical Sciences, Washington 25, D. C.

4. *Office of Ordnance Research, U. S. Army.* Among the functions of the Office of Ordnance Research is the support of basic research in mathematics. Proposals for projects are ordinarily made by individual scientists or groups of scientists in a form which leads to a contract between the Office of Ordnance Research and a university or research laboratory. For further information write to Commanding Officer, Office of Ordnance Research, Box CM, Duke Station, Durham, North Carolina.

5. *Fulbright Awards—Public Law 584 (79th Congress).* Approximately 400 awards are offered annually for university lecturing and postdoctoral research in all academic fields in Argentina, Australia, Brazil, Burma, Chile, Colombia, Ecuador, India, New Zealand, Pakistan, Paraguay, Peru, the Philippines and Thailand (the next competition for the preceding countries, for the 1961–62 academic year, closes April 25, 1960); Austria, Belgium-Luxembourg, Republic of China, Denmark, Finland, France, Germany, Greece, Iceland, Iran, Ireland, Israel, Italy, Japan, the Netherlands, Norway, Spain, Turkey, and the United Kingdom including colonial dependencies (competition for the latter countries, for the 1960–61 academic year, closes October 1, 1959). 1960–61 lectureships in mathematics are offered in the Republic of China, Finland, Ireland, Japan, Singapore, and the United Kingdom. Awards are payable in foreign currency and usually include travel for the grantee, but not for members of his family, and a maintenance allowance, which may be adjusted in relation to the number of accompanying dependents up to four. Requests for information should be addressed to the Committee on International Exchange of Persons, Conference Board of Associated Research Councils, 2101 Constitution Avenue, Washington 25, D. C.

6. *National Bureau of Standards. Naval Research Laboratory. Air Research and Development Command. Naval Ordnance Laboratory. Navy Electronics Laboratory.* These highly desirable appointments for postdoctoral study and research are similar to postdoctoral fellowships. Postdoctoral resident research associateships are available in a variety of sciences including mathematics and are tenable at the Washington, D. C. and Boulder, Colorado, laboratories of the National Bureau of Standards; at the Naval Research Laboratory in Washington, D. C.; at selected development and research centers of the Air Research and Development Command; at the Naval Ordnance Laboratory in Silver Spring, Maryland; and at the Naval Electronics Laboratory in San Diego, California.

7. *Atomic Energy Commission.* The Division of Research of the Atomic Energy Commission through contracts with universities and other organizations supports research in the fields of numerical analysis, digital computer design, programming research, and related topics. Proposals should be submitted to the Division of Research, Atomic Energy Commission, Washington 25, D. C.

Brookhaven National Laboratory. Brookhaven National Laboratory, operated by Associated Universities, Inc. under contract with the Atomic Energy Commission offers postdoctoral research appointments in the fields of numerical analysis, digital computing, mathematical physics, differential equations, probability and statistics, and various specialized branches including reactor theory, hydrodynamics, and orbit theory. Applications should be directed to M. E. Rose, Head, Applied Mathematics Division, Brookhaven National Laboratory, Upton, Long Island, New York.

MEETING OF MATHEMATICS SECTION—AAAS

The American Association for the Advancement of Science will meet December 26–31 in Chicago. Section A (Mathematics) will have three meetings. On December 26 at 4 P.M. in the Promenade Room of the Morrison Hotel, R. H. Bing will give his retiring vice-presidential address "Topology of Euclidean Three-space."

On December 27 at 9 A.M. in the Promenade Room of the Morrison Hotel there will be four invited papers on "The New Look in Mathematical Education":

G. Baley Price, the University of Kansas and the California Institute of Technology, "Report on the Work of the Committee on the Undergraduate Program."

Henry Swain, Winnetka, Illinois, "The Ninth Grade Course of the School Mathematics Study Group."

Morris Kline, New York University, "New Curriculum or New Pedagogy?"

William L. Duren, Jr., University of Virginia, "Early Calculus in the United States."

On December 28 at 9 A.M. in Parlor F of the Morrison Hotel there will be a symposium on "Trends in the Applications of Mathematics" cosponsored by Section A and the Society for Industrial and Applied Mathematics:

R. F. Drenick, Bell Telephone Laboratories, "Probability and Stochastic Processes."

Philip Wolfe, Rand Corporation, "Mathematical Programming."

Frank Wagner, North American Airplane Company, "Computers and Computer Languages."

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Treasurer, announces that the following 144 persons have been elected to membership by the Board of Governors on applications duly certified.

ALBERT W. ALEXANDER, A.B. (Atlantic Christian) Teacher, Contentnea High School, Kinston, North Carolina.

HOWARD M. AMEN, M.E. (South Dakota) Instr., South Dakota State College.

THOMAS A. ATCHISON, B.S. (Texas) Grad. Student, University of Texas.

MICHAEL W. ATKINSON, B.S. (Ball S.T.C.) Grad. Student, Ball State Teachers College.

MRS. KEMP H. BALDWIN, M.A. (East Carolina) Teacher, J. H. Rose High School, Greenville, North Carolina.

MARY E. BEAUMONT, M.S. (Wisconsin) Instr., Ripon College.

ALBERT F. BENINATI, M.A. (N.Y.S. T.C., Albany) Math. Supervisor, Nyack High School, New York.

CHARLES A. BERGREN, M.A. (Columbia) Engineer, Airborne Instruments Lab.

GLORIA A. BERNIER, B.S. (St. Lawrence) Research Mathematician, St. Regis Paper Co.

DEBORAH W. BEVERIDGE, M.A.T. (Radcliffe) Teacher, Milton Academy, Massachusetts.

GERALD G. BILODEAU, Ph.D. (Harvard) Advanced Research Engineer, Sylvania Electronic Systems.

HAMILTON S. BLUM, Prof. diploma (Columbia) Chairman of Dept., East Meadow High School, New York.

LEONIE D. BOEHMER, B.S. (Coll. of St. Joseph) Calculating Machine Operator, Fox Vliet Drug Co.

PETER M. BRADY, JR., Student, Stevens Institute of Technology.

LOUISE BRAKE, M.A. (East Carolina) Teacher, Snow Hill High School, North Carolina.

OSCAR W. BRANNON, A.B. (Atlantic Christian) Teacher, Windsor High School, North Carolina.

A. MARTIN BUONCRISTIANI, Student, University of Santa Clara.

H. BLAIR BURNER, JR., M.S. (North Dakota) Instr., University of North Dakota.

JACK F. BUSSIO, A.A. (Sacramento J.C.) Student, University of California, Davis.

DAVID G. CANTOR, B.S. (California Tech.) NSF Fellow, University of California, Los Angeles.

MARION E. CLARK, M.S. (Michigan) Asso. Professor, King College.

EILEEN C. COX, A.M. (Columbia) Acting Chairman of Dept., New Dorp High School, Staten Island.

CHARLES R. DAVIS, B.S. (East Carolina) Teacher, Morehead City School, North Carolina.

FELIX T. DAVIS, III, Student, University of Kansas.

JOHN B. DAVIS, JR., M.A. (East Carolina Coll.) Asso. Professor, Wilmington College.

LOUIS E. DENOYA, Student, Oklahoma State University.

EARL W. DENTON, M.A. (East Carolina) Teacher, Grifton High School, North Carolina.

MAURICE C. DEVORE, B.S. (Nevada) James Marshall High School, W. Sacramento, California.

EDWARD D. EBERT, M.S. (Iowa) Asst. Professor, University of Toledo.

LOUIS M. EDWARDS, M.Ed. (Florida) Teacher, Edgewater High School, Orlando, Florida.

MARY ELIZABETH EKSTRAND, B.A. (California, Davis) Mathematician, Naval Ordnance Test Station.

SAMUEL FEDER, M.S. (New York) Mathematician, Bulova Research & Development Labs.

ENRICO T. FEDERIGHI, Ph.D. (Indiana) Sr. Mathematician, Applied Physics Lab.

KIMBROUGH S. GARRETT, Student, University of Texas.

ELDRIDGE W. GARTEN, B.A. (Shelton Coll.) Engineering Change Analyst, Republic Aviation Corp.

RALPH H. GOEBEL, M.S. (S.U. of Iowa) Asso. Actuary, Northwestern National Life Insurance Co.

GLENN W. GOODRICH, Student, University of Tulsa.

EDMOND B. GORMAN, B. S. in E.E. (Healds Eng.) Design Engineer, Federal Pacific Electric Co.

GEORGE C. GRAFF, B.S. (Brooklyn Coll.) Grad. Asst., University of Illinois.

JACK S. GRAY, B.S. (East Carolina) Asst. Principal, Gar-Field High School, Woodbridge, Virginia.

MRS. BETTY S. GUILFORD, B.S. (Appalachian S.T.C.) Teacher, Chocowinity High School, North Carolina.

ROBERT A. HALL, M.S. (Washington) Instr., University of Minnesota, Duluth.

EVA C. HANNEMAN, A.M. (Cornell) Teacher, Waverly Senior High School, New York.

MERTEN M. HASSE, M.A. (Carleton Coll.) Asst. Professor, State University of South Dakota.

ALVIE L. HASTE, B.S. in Ed. (Ohio State) Teaching Asst., University of Cincinnati.

COL. N. L. HEAD, Ed.M. (Boston) Teacher, Lexington High School, Massachusetts.

JOSEPH A. HEBERT, Student, Bradford Durfee College of Technology.

HELEN E. HOBBIE, M.A. (N.Y.S.T.C., Albany) Teacher, Bethlehem Central Senior High School, Delmar, New York.

ALBERT H. HOFFMAN, JR., B.A. (Florida) Grad. Student, University of Florida.

ANNE HONDELINK, Ed.M. (Rochester) Head of Dept., Brighton High School, Rochester, New York.

CONRAD O. HOPPERSTEAD, B.S. (Texas) Computing Analyst, Temco Aircraft Corp.

CLAUDIA M. HOUSE, A.B. (Bucknell) Mathematician, General Electric Co.

EUGENE M. HUGHES, M.S. (Kansas S.U.) Head of Dept., Nebrsaka State Teachers College.

STANLEY S. ISAACS, Student, Antioch College.

ARNOLD A. JOHNSON, Ph.D. (Case Inst. of Tech.) Project Engineer, Case Institute of Technology.

MRS. EMMA D. JOHNSON, M.A. (Ohio State) Asst. Professor, Ohio Wesleyan University.

HARLEY R. JORDAN, B.S. (Denver) Physicist, Navy Department.

JOANNE M. KAMINSKY, B.S. (Harpur) Mathematician, International Business Machines.

HAROLD M. KAPLAN, M.A. (Princeton) Asst. Professor, United States Naval Academy.

MAURICE KAPLOWITZ, M.A. (New York) Teacher, Boys High School, Brooklyn, New York.

HENRY C. KASHIAN, M.A. (Boston) Sr. Engineer, Sylvania Electric Products.

GORDON E. KELLER, Student, Houghton College.

SIDNEY KELLMAN, B.A. (Geo. Washington) Mathematician, Naval Air Material Center.

MRS. DOROTHY A. KENNEDY, M.A. (Columbia) Teacher, Sweet Home Central High School, Amherst, New York.

RICHARD M. KENNEDY, M.A. (Massachusetts) Instr., University of Massachusetts.

GEORGE F. KIENLEN, M.A. (Colorado) Teacher, South High School, Denver, Colorado.

JOHN J. KIM, Student, Eastern New Mexico University.

KENNETH E. KNOX, M.S. (Washington) Parsons Junior College.

DEENA ANN KONIVER, Student, Massachusetts Institute of Technology.

HARVEY N. LANCE, M.Ed. (Furman) Professor, Mars Hill College.

RUBY LANGFORD, M.A. (East Carolina) Teacher, Smithfield High School, North Carolina.

GEORGE D. LAWRENCE, Student, Hobart College.

JANE LEE, A.B. (Indiana) Engineering Asst., General Electric Co.

DANA J. LEFSTAD, M.S. (Wisconsin) Teacher, Skagit Valley College.

MRS. M. BROOKE LEONARD, Student, Montana State University.

MARIAN A. LESHER, M.A. (Colorado S.C.) Asst. Professor, Central Missouri State College.

MAYER LEVINE, B.S. (Houston) Mathematician, Naval Ordnance Lab.

DAVID A. LEVINSON, B.A. (Temple) Supervisor, Computer Lab., University of Cincinnati.

WILLIAM A. LIVINGSTON, JR., B.A. (Buffalo) Asst. Physicist, Cornell Aeronautical Lab.

MORTON LOWENGRUB, M. S. (California Inst. Tech.) Part-time Instr., Duke University.

ROBERT E. LOWNEY, Ph.D. (Wisconsin) Professor, Montana State College.

ARSETE J. LUCCHESI, M.S. (New York) Engineer, Eclipse-Pioneer, Teterboro, New Jersey.

GLEN LUCHAU, Student, University of Oklahoma.

JOHN B. MAJOR, M.A. (Columbia) Chairman of Dept., Freeport Senior High School, New York.

W. JAMES MATHERS, B.A. (McMaster) Head of Dept., Medway High School, London, Ontario.

TED O. MCCARLEY, B.S. (East Central S.C.) Grad. Student, University of Notre Dame.

HUBERT MCGEE, JR., B.S. (East Carolina) Teacher & Chairman of Dept., New Bern High School, North Carolina.

CHRISTOPHER M. MCGUIRE, Student, University of Arizona.

DONALD H. MCINNIS, M.A. (Missouri) Operations Analyst, McDonnell Aircraft Corp.

ANN E. MCKINNON, A.B. (Salem Coll.) Teacher, Maxton High School, North Carolina.

MRS. MARY S. MERCER, A.B. (Winthrop Coll.) Head of Dept., Beulaville High School, North Carolina.

SAMUEL MERRILL, III, Student, Tulane University.

JOHN R. MESSINGSCHLAGER, Student, University of Detroit.

URSULA MRAZEK, Student, University of Alabama.

M. ELIZABETH NAILOS, A.B. (Wm. Smith Coll.) Teacher, Waterloo Central School, New York.

ROBERT E. R. NELSON, B.A. (Virginia) Grad. Student, University of Virginia.

ANTHONY J. NESPOLE, M.A. (C.C. N.Y.) Lecturer, City College of New York.

MRS. KATHLEEN B. O'KEEFE, Ph.D. (California) Instr., Hunter College.

ROBERT L. PAGE, B.S. (Tuft) Grad. Asst., University of Maine.

REV. JOSEPH E. PASZEK, S.J., B.A. (Fordham) Teacher, St. Joseph's College High School, Philadelphia, Pennsylvania.

RICHARD F. PAVLEY, M.A. (Wayne S.U.) Part-time Instr., Syracuse University.

MRS. FLORENCE C. PISANO, M.A. (Columbia) Teacher, Board of Education, Mount Vernon, New York.

MELVIN L. POAGE, M.B.S. (Colorado) Teacher, South High School, Denver, Colorado.

IVAN P. POLONSKY, Ph.D. (New York) Instr., Queens College.

JACK O. PURDUE, Ph.D. (Oklahoma) Chairman, Div. of Natural Sciences, Oklahoma Baptist University.

SYLVESTER REESE, B.S. (Morgan S.C.) Grad. Student, University of Wisconsin.

WALLACE S. REID, M.S. (Wisconsin) Instr., Upsala College.

JOHN L. ROBERSON, B.S. (East Carolina) Teacher, Robersonville High School, North Carolina.

JOSEPH A. ROSS, M.Ed. (Pittsburgh) Teacher, Aliquippa Public Schools, Pennsylvania.

RICHARD E. RUSSELL, A.B. (Temple) Instr., Drexel Institute of Technology.

ELLIS W. SCHONER, M.A. (Akron) Teacher, U.L. Light Junior High School, Barberton, Ohio.

MRS. EVELYN E. SEDAR, M.A. (Wyoming) Instr., Casper College.

GERALD S. SHEDLER, Student, Amherst College.

FRANKLIN F. SHEEHAN, B.S. (Stanford) Asso. Professor, United States Naval Postgraduate School.

OVED SHISHA, Ph.D. (Hebrew) Research Asso., Harvard University.

SISTER HELEN LOUISE, O.P., M.A. (Michigan) Head of Dept., Aquinas College.

SISTER RITA JEAN, C.S.J., M.A. (Minnesota) The College of St. Catherine.

GEORGE F. SMITH, M.S. (Massachusetts) Teacher, South Hadley High School, Massachusetts.

JOAN SPIGNER, B.S. (Newberry Coll.) Teacher, York High School, South Carolina.

HARRY L. STEIN, Ph.D. (Minnesota) Supervisor of Grad. Studies and Math. Lecturer, University of British Columbia.

THORNTON G. STOVALL, A.B. (East Carolina) Teacher, Stovall High School, North Carolina.

RALPH E. STRAUCH, A.B. (California, Los Angeles) Ensign, United States Navy.

GRACE STRECKER, M.A. (St. Louis) Teacher, Normandy High School, St. Louis County, Missouri.

ROBERT L. SVEHLA, B.A. (Doane Coll.) Reliability Analyst, Boeing Airplane Co.

FLOYD L. TAYLOR, Ph.D. (Nebraska) Chairman of Dept., Southern Oregon College.

FOUNTAIN TAYLOR, JR., B.S. (East Carolina) Teacher, Jacksonville High School, North Carolina.

MAYNARD D. THOMPSON, M.S. (Wisconsin) 207-G Eagle Heights, Madison 5, Wisconsin.

ELIZABETH A. TOBIN, B.A. (Connecticut) Part-time Instr., University of Conecticut.

FLETCHER P. TOMIC, A.B. (Harvard) Teacher, Technical High School, Buffalo, New York.

JOHN L. TRAUB, M.S. (Wisconsin) Instr., Wisconsin State College.

BRYANT TRIPP, A.B. (Elon Coll.) Teacher, Maury High School, North Carolina.

ANTHONY P. TRUJILLO, M.A. (Denver) Instr., Mohawk Valley Technical Institute, Utica, New York.

HOWARD H. W. UN, Student, Beloit College.

MILTON M. UNDERKOFFLER, M.S. (Illinois S.N.U.) Instr., Winona State College.

THOMAS O. VINSON, JR., Student, Emory University.

MRS. EARLINE Y. WEBB, M.A. (Sam Houston S.T.C.) Teacher, Box 177, Teague, Texas.

WALTER G. WESLEY, Student, Texas Christian University.

MRS. SONIA L. WEST, A.B. (East Carolina) Teacher, Charles L. Coon Junior High School, Wilson, North Carolina.

MARY ALYCE WILLIAMS, M.A. (East Carolina) Teacher, Aurelian Springs School, Littleton, North Carolina.

DAVID L. WOOD, B.S. (Miami) Teacher, Coral Gables Senior High School, Florida.

LT. JIMMIE D. WOODS, B.S. (U.S. Coast Guard Acad.) U.S. Coast Guard Base, San Juan, Puerto Rico.

RUBIN P. WOOTEN, Student, Mississippi State University.

RONALD L. WRIGHT, Student, Santa Ana College.

JUDITH ZAGRODNICK, B.S. (Pittsburgh) Sr. Technical Asst., Bell Telephone Labs.

THE FORTIETH SUMMER MEETING OF THE ASSOCIATION

The fortieth summer meeting of the Mathematical Association of America was held at the University of Utah, Salt Lake City, Utah, from Monday, August 31 through Thursday, September 3, 1959 in conjunction with summer meetings of the American Mathematical Society, the Association for Symbolic Logic, the Society for Industrial and Applied Mathematics, Pi Mu Epsilon, and Mu Alpha Theta. There were registered 985 persons, including 423 members of the Association.

Sessions of the MAA were held on Monday morning and afternoon, on Tuesday and Wednesday mornings and on Thursday afternoon. All sessions were held in Spencer Hall Auditorium of the University of Utah. Presiding officers were President C. B. Allendoerfer, Vice-president Harley Flanders, and Professors C. B. Tompkins, A. Nijenhuis, R. F. Rinehart and F. E. Hohn. The eighth series of Earle Raymond Hendrick Lectures were delivered by Professor William Feller of Princeton University. The Program Committee for the meeting consisted of J. V. Wehausen, Chairman; G. E. Forsythe, F. E. Hohn, A. Nijenhuis, and R. F. Rinehart.

FIRST SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: "Topics Connected with Ordinary Differential Operators," Lecture I, by Professor William Feller, Princeton University.

"Training of High School Mathematics Teachers," by Professor J. L. Kelley, University of California, Berkeley. Panel: C. B. Allendoerfer, E. G. Begle, Harley Flanders, L. C. Lay.

"The Role of Films and Television in Teaching Mathematics," by Professor H. M. MacNeille, Washington University.

SECOND SESSION OF THE ASSOCIATION

Hedrick Lecture II, by Professor Feller.

"High Speed Computing in a University," Chairman, Professor C. B. Tompkins, University of California, Los Angeles.

"Recent Developments in Computer Languages," by Professor Bernard A. Galler, University of Michigan.

"Organizing a University Computation Center," by Professor J. G. Herriot, Stanford University.

"The Acquisition and the Role of the Digital Computer in Research," by Dean Henry Eyring, University of Utah.

THIRD SESSION OF THE ASSOCIATION

Hedrick Lecture III, by Professor Feller.

Business Meeting of the Association.

"The Role of Geometry for the Mathematics Student," by Professor Herbert Busemann, University of Southern California.

"Geometry for Teachers in the Undergraduate Curriculum," by Professor E. E. Moise, University of Michigan.

FOURTH SESSION OF THE ASSOCIATION

"Stimulations to Mathematics from Operations Research," Chairman, Professor R. F. Rinehart, Duke University.

"Mathematical Programming," by Dr. Robert Kalaba, Rand Corporation.

"Stochastic Processes; Queuing Theory," by Dr. Donald P. Gaver, Jr., Westinghouse Electric Corporation (by title).

"Calculus of Variations," by Professor J. M. Danskin, Rutgers University.

FIFTH SESSION OF THE ASSOCIATION

Retiring Presidential Address: "Analytic Functions of a Bicomplex Variable," by Professor G. B. Price, University of Kansas.

"Theory of Digital Circuits," Chairman, Professor F. E. Hohn, University of Illinois.

"Some Applications of Logic to the Theory of Sequential Circuits," by Professor F. B. Fitch, Yale University.

"The Logico-Mathematical Theory of Synchronous Circuits," by Dr. C. C. Elgot, Willow Run Laboratories, University of Michigan.

"The Mathematical Theory of Asynchronous Circuits," by Professor D. E. Muller, University of Illinois.

SPECIAL SESSIONS OF THE ASSOCIATION

The set of four films on "The Theory of Limits" starring E. J. McShane and produced by the MAA's Committee on Production of Films was shown in Kingsbury Hall Auditorium on Wednesday evening. The first two films were also shown in Spencer Hall on Thursday afternoon.

On Thursday evening an open conference on High School Contests was held in the Union Building, with about 25 persons present. President Allendoerfer presided. There was general agreement that the contest should be continued along its present lines with some suggested modifications in the nature of the questions.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Monday evening in the Union Building of the University of Utah with twenty-one members present. Among the more important items of business transacted were the following:

Professor R. D. Wagner of the University of Wisconsin was elected to the Board of Governors to fill the balance of the unexpired term of the late Professor H. P. Evans.

The Board approved the following schedule of future meetings: Hotel Conrad Hilton, Chicago, Illinois, January 28–30, 1960; Michigan State University, August 29–September 1, 1960; Hotel Willard, Washington, D. C., January, 1961; Oklahoma State University, August, 1961; Kansas City, Missouri, January, 1962; University of British Columbia, August, 1962; University of Colorado, August, 1963.

The Board voted to invite Professor Ivan Niven of the University of Oregon, to deliver the ninth series of Earle Raymond Hedrick Lectures at the 1960 Summer Meeting.

Publication was authorized of Carus Monograph No. 13 entitled "A Primer of Real Functions" by R. P. Boas.

Action was taken to increase the size of the MONTHLY so that the normal issue hereafter will consist of 112 pages rather than 96 pages. This will provide for more rapid publication of accepted papers.

The Board voted to make an appropriation for the support of the Mathematics Magazine in order to provide for its continued publication.

BUSINESS MEETING OF THE ASSOCIATION

A business meeting of the Association was held on Thursday morning with President Allendoerfer presiding. The Secretary-Treasurer reported that the membership of the Association was 8633 on August 17, an increase of 11% since the corresponding date last year. Over 10,000 copies of the MONTHLY are being printed for each issue.

Professor Henry L. Alder, the new Secretary of the Association was then presented to the meeting. He expressed the appreciation of the members of the Association for the services of Professor Gehman as Secretary-Treasurer.

Professor Rothwell Stephens reported for the Committee on Visiting Lecturers and Dr. John R. Mayor reported for the Committee on Secondary School Lecturers.

MEETING OF SECTION OFFICERS

A meeting of representatives of the Sections of the Association was held on Tuesday evening in the Union Building. Forty-eight persons were present representing 25 of the 27 Sections of the Association.

Reports were made on special activities of the various sections, such as speakers bureaus, traveling lecturers programs, and committees on high school-college relations. There was also a general discussion of programs of section meetings.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held its sessions from Tuesday afternoon through Friday. The colloquium speaker was Professor J. L. Doob and invited addresses were given by Professors P. A. Smith and E. A. Coddington.

The Association for Symbolic Logic met on Friday. SIAM met from Tuesday evening through Friday morning.

Pi Mu Epsilon held a luncheon meeting on Tuesday. Mu Alpha Theta, the national high school and junior college mathematics club, held a luncheon meeting on Wednesday.

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements for the meeting consisted of: W. J. Coles, Chairman; J. H. Barrett, Lida K. Barrett, F. C. Biesele, J. H. Curtiss, H. M. Gehman, I. O. Horsfall, N. C. Hunsaker, E. E. Kohlbecker, R. S. Pierce, D. W. Robinson, C. R. Wylie.

Registration headquarters was located in the Union Building. Dormitory and cafeteria accommodations were provided by the University of Utah. The text book exhibit and the Mathematical Sciences Employment Register were located in the Union

Building. The Register was augmented by a Placement Service consisting of listings of individuals available for positions.

An informal reception for the women attending the meeting was given by the wives of the Department of Mathematics, University of Utah, on Tuesday afternoon in Carlson Hall Lounge. A picnic at Brighton at the head of Big Cottonwood Canyon was held on Wednesday afternoon. The usual SIAM social evening was held on Wednesday evening. A musical program was presented on Thursday evening at Temple Square especially for persons attending the meetings.

A resolution of thanks prepared by Professor H. M. MacNeille and adopted by the participating organizations expressed warm appreciation to our host, the University of Utah, for all that it has contributed to make our stay here both pleasant and memorable. Thanks was also expressed to the University staff and in particular to the members of the local Committee on Arrangements for so ably coordinating the many phases of the arrangements which have contributed so much to our comfort and enjoyment.

HARRY M. GEHMAN, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Forty-third Annual Meeting, Conrad Hilton Hotel, Chicago, Illinois, January 28–30, 1960.

Forty-first Summer Meeting, Michigan State University, East Lansing, Michigan, August 29-September 1, 1960.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Grove City College, Grove City, Pennsylvania, April 30, 1960.

ILLINOIS, Illinois Wesleyan University, Bloomington, May 13–14, 1960.

INDIANA

IOWA, State University of Iowa, Iowa City, April 22, 1960.

KANSAS, Kansas State College of Pittsburg, April 30, 1960.

KENTUCKY, University of Kentucky, Lexington, April, 1960.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 19–20, 1960.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, American University, Washington, D. C., December 5, 1959.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March 26, 1960.

MINNESOTA

MISSOURI, Central Missouri State College, Warrensburg, April 30, 1960.

NEBRASKA, University of Nebraska, Lincoln, April 23, 1960.

NEW JERSEY, Princeton University, November 7, 1959.

NORTHEASTERN, Boston College, Chestnut Hill, Massachusetts, November 28, 1959.

NORTHERN CALIFORNIA, University of California, Berkeley, January 16, 1960.

OHIO, Kent State University, May 7, 1960.

OKLAHOMA

PACIFIC NORTHWEST, State University of Montana, Missoula, June 17, 1960.

PHILADELPHIA, University of Delaware, Newark, November 28, 1959.

ROCKY MOUNTAIN, United States Air Force Academy, Colorado Springs, May 6–7, 1960.

SOUTHEASTERN, University of South Carolina, Columbia, April 1–2, 1960.

SOUTHERN CALIFORNIA, Los Angeles State College, March 12, 1960.

SOUTHWESTERN, Air Force Missile Development Center, Holloman Air Force Base, New Mexico, April, 1960.

TEXAS, San Antonio College, April, 1960.

UPPER NEW YORK STATE, University of Rochester, May 7, 1960.

WISCONSIN, Mount Mary College, Milwaukee, May 7, 1960.

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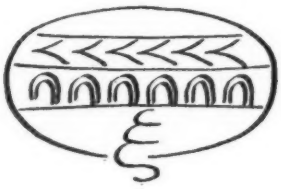

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